Chapter 13 Scalar Products and Vector Products
Supplementary Notes

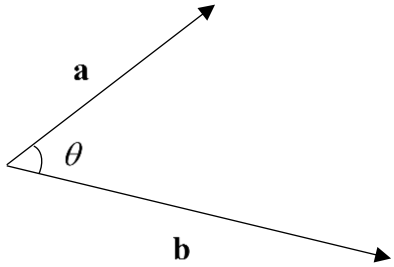
Name: ______________________________ (   ) Class: F.5 _____

13.1 Scalar Products

A. Definition of Scalar Product

In this section, we are going to introduce another operation of vectors called the **scalar product**, which is related to a kind of multiplication operation of two vectors.

θ is the angle between two non-zero vectors **a** and **b** when the initial points of two vectors coincide, where $0 \le \theta \le 180°$.



Definition of Scalar Product:

If **a** and **b** are non-zero vectors, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

If **a** or **b** is a zero vector, then $\mathbf{a} \cdot \mathbf{b} = 0$.

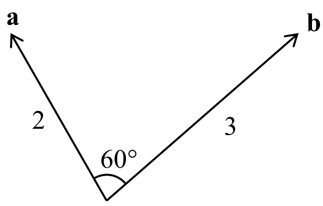
Remarks:

1. The scalar product is also called the **dot product** and $\mathbf{a} \cdot \mathbf{b}$ is read as '**a** dot **b**'.
2. Since $|\mathbf{a}|$ and $|\mathbf{b}|$ are the magnitudes of vectors **a** and **b** respectively, $\mathbf{a} \cdot \mathbf{b}$ is a **scalar (a number)**, not a vector.
3. $\mathbf{a} \cdot \mathbf{b}$ cannot be written as $\mathbf{ab}$ or $\mathbf{a} \times \mathbf{b}$.

Example

1. Find $\mathbf{a} \cdot \mathbf{b}$ in the following.

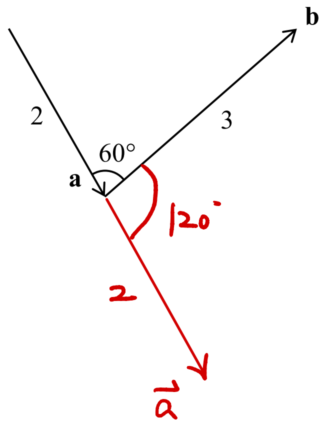
(a)



$\vec{a} \cdot \vec{b} = 2 \cdot 3 \cdot \cos 60°$

$= 3$

(b)



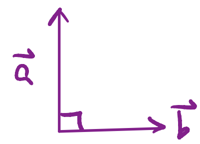
$\vec{a} \cdot \vec{b}$

$= 2 \cdot 3 \cdot \cos 120°$

$= -3$

*2. (a) Find the value of $\mathbf{a}\cdot\mathbf{b}$ if $\mathbf{a}\perp\mathbf{b}$. $\theta = 90^\circ$

(b) Express $\mathbf{a}\cdot\mathbf{a}$ in terms of $|\mathbf{a}|$. $\theta = 0^\circ$



(a) $\vec{a}\cdot\vec{b}$

$= |\vec{a}|\cdot|\vec{b}|\cdot \cancel{\cos 90^\circ}^{\,0}$

$= 0$

(b) $\vec{a}\cdot\vec{a}$

$= |\vec{a}|\cdot|\vec{a}|\cdot\cos 0^\circ \; {=1}$

$= |\vec{a}|^2$

B. Properties of Scalar Product

For any vectors **a**, **b**, **c** and scalar λ,

(a) $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 \geq 0$ (***Commonly used to find the magnitude of **a**)

(b) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

(c) $\mathbf{a}\cdot\mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{0}$

(d) $\lambda(\mathbf{a}\cdot\mathbf{b}) = (\lambda\mathbf{a})\cdot\mathbf{b} = \mathbf{a}(\lambda\mathbf{b})$

(e) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$

(f) $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$

(g) $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(\mathbf{a}\cdot\mathbf{b})$

(g) $|\vec{a}-\vec{b}|^2$

$= (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}\cdot\vec{b})$

Example

3. It is given that $|\mathbf{a}| = 2\sqrt{3}$, $|\mathbf{b}| = 6$ and the angle between **a** and **b** is $30°$. Find the value of the following.

(a) $\mathbf{a} \cdot \mathbf{b}$

(b) $\mathbf{a} \cdot (\mathbf{a} - 2\mathbf{b})$

(c) $|\mathbf{a} + 2\mathbf{b}|^2$

(a) $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos 30°$

$= 2\sqrt{3} \cdot 6 \cdot \frac{\sqrt{3}}{2}$

$= 18$

(b) $\vec{a} \cdot (\vec{a} - 2\vec{b})$

$= \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b}$

$= |\vec{a}|^2 - 2 \cdot 18$

$= 12 - 36$

$= -24$

(c) $|\vec{a} + 2\vec{b}|^2$

$= (\vec{a} + 2\vec{b}) \cdot (\vec{a} + 2\vec{b})$

$= \vec{a} \cdot \vec{a} + 4\vec{a}\vec{b} + 4\vec{b} \cdot \vec{b}$

$= |\vec{a}|^2 + 4 \cdot 18 + 4|\vec{b}|^2$

$= 12 + 72 + 144$

$= 228$

4. It is given that $|\mathbf{a}| = 2\sqrt{2}$, $|\mathbf{b}| = 4$ and the angle between **a** and **b** is $45°$. Find the value of the following.

(a) $\mathbf{a} \cdot \mathbf{b}$

(b) $\mathbf{a} \cdot (\mathbf{b} + 2\mathbf{a})$

(c) $|\mathbf{a} - \mathbf{b}|^2$

(a) $\vec{a} \cdot \vec{b}$

$= |\vec{a}| \cdot |\vec{b}| \cdot \cos 45°$

$= 2\sqrt{2} \cdot 4 \cdot \frac{\sqrt{2}}{2}$

$= 8$

(b) $\vec{a} (\vec{b} + 2\vec{a})$

$= \vec{a} \cdot \vec{b} + 2\vec{a} \cdot \vec{a}$

$= 8 + 2|\vec{a}|^2$

$= 8 + 16$

$= 24$

(c) $|\vec{a} - \vec{b}|^2$

$= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$

$= \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$

$= |\vec{a}|^2 - 2 \cdot 8 + |\vec{b}|^2$

$= 8 - 16 + 16$

$= 8$

5. It is given that **u** and **v** are unit vectors and angle between the vectors is $60°$.

(a) Find $\mathbf{u}\cdot\mathbf{v}$.

(b) Find $(\mathbf{u}-2\mathbf{v})\cdot(4\mathbf{u}+\mathbf{v})$.

$|\vec{u}| = |\vec{v}| = 1$

* (c) Find $|2\mathbf{u}+\mathbf{v}|$. → Find $|2\vec{u}+\vec{v}|^2$, take $+\sqrt{\ }$

(a) $\vec{u}\cdot\vec{v}$

$= |\vec{u}|\cdot|\vec{v}|\cdot\cos 60°$

$= 1\cdot 1\cdot \frac{1}{2}$

$= \frac{1}{2}$

(b) $(\vec{u}-2\vec{v})\cdot(4\vec{u}+\vec{v})$

$= 4|\vec{u}|^2 - 7\vec{u}\cdot\vec{v} - 2|\vec{v}|^2$

$= -\frac{3}{2}$

(c) $|2\vec{u}+\vec{v}|^2 = (2\vec{u}+\vec{v})\cdot(2\vec{u}+\vec{v})$

$= 4|\vec{u}|^2 + 4\vec{u}\cdot\vec{v} + |\vec{v}|^2$

$= 7$

$\therefore |2\vec{u}+\vec{v}| = \sqrt{7}$

C. Applications of Scalar Products

I. Angle between Vectors

Consider two non-zero vectors **a** and **b**. Let θ be the angle between **a** and **b**.

$\because \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, we have $\boxed{\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}}$

<u>Example</u>

6. **a** and **b** are two vectors such that $|\mathbf{a}|=1$, $|\mathbf{b}|=2$ and $\mathbf{a}\cdot\mathbf{b}=-1$.

(a) Find $|\mathbf{a}+3\mathbf{b}|$.

(b) Find the angle between **b** and $\mathbf{a}+3\mathbf{b}$.

(Give your answers correct to the nearest $0.1°$)

(a) $|\vec{a}+3\vec{b}|^2$

$= (\vec{a}+3\vec{b})\cdot(\vec{a}+3\vec{b})$

$= |\vec{a}|^2 + 6\vec{a}\cdot\vec{b} + 9|\vec{b}|^2$

$= 1 - 6 + 36$

$= 31$

$|\vec{a}+3\vec{b}| = \sqrt{31}$

(b) $\vec{b}\cdot(\vec{a}+3\vec{b}) = \vec{a}\cdot\vec{b} + 3|\vec{b}|^2$

$= 11$

required angle $= \cos^{-1}\left(\dfrac{11}{2\cdot\sqrt{31}}\right)$

$= 8.9°$

7. **a** and **b** are two vectors such that $|\mathbf{a}|=5$, $|\mathbf{b}|=4$ and $\mathbf{a}\cdot\mathbf{b}=3$. Find the angles between

(a) **a** and **b**;

(b) 5**a** and $\mathbf{a}-\mathbf{b}$.

(*Give your answers correct to the nearest* $0.1°$)

(a) required angle

$= \cos^{-1} \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

(b) $5\vec{a}\cdot(\vec{a}-\vec{b})$

$= 5|\vec{a}|^2 - 5\vec{a}\cdot\vec{b}$

$= 110$

$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-b)$

$= |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$

$= 35$

$|\vec{a}-\vec{b}| = \sqrt{35}$

required angle $= \cos^{-1} \dfrac{110}{5\times 5\times\sqrt{35}}$

$= 41.9°$

II. Orthogonality

If the angle between two non-zero vectors **a** and **b** is $90°$, then the two vectors are perpendicular to each other (i.e. $\mathbf{a}\perp\mathbf{b}$) and they are called orthogonal **vectors**.

For non-zero vectors **a** and **b**, $\mathbf{a}\cdot\mathbf{b}=0$ if and only if $\mathbf{a}\perp\mathbf{b}$.

Example

8. It is given that **a** and **b** are two vectors, where $|\mathbf{a}|=2$, $|\mathbf{b}|=3$ and the angle between **a** and **b** is $60°$.

(a) Find $\mathbf{a}\cdot\mathbf{b}$.

(b) If $\mathbf{u}=\mathbf{b}-\mathbf{a}$ and $\mathbf{v}=6\mathbf{a}+\mathbf{b}$, show that **u** and **v** are orthogonal.

(a) $\vec{a}\cdot\vec{b}$

$= |\vec{a}|\cdot|\vec{b}|\cdot\cos 60°$

$= 2\cdot 3\cdot\frac{1}{2}$

$= 3$

(b) $\vec{u}\cdot\vec{v}$

$= (\vec{b}-\vec{a})\cdot(6\vec{a}+\vec{b})$

$= 6\vec{a}\cdot\vec{b} + |\vec{b}|^2 - 6|\vec{a}|^2 - \vec{a}\cdot\vec{b}$

$= 5\cdot 3 + 9 - 6\cdot 4$

$= 0$

9. $\mathbf{u} = 3\mathbf{a} - \mathbf{b}$ and $\mathbf{v} = 2\mathbf{a} + 4\mathbf{b}$ are orthogonal vectors, where $|\mathbf{a}| = 2\sqrt{6}$ and $|\mathbf{b}| = 6$.

Prove that **a** and **b** are orthogonal vectors.

$\vec{u} \cdot \vec{v} = 0$

$(3\vec{a} - \vec{b}) \cdot (2\vec{a} + 4\vec{b}) = 0$

$6|\vec{a}|^2 + 12\vec{a} \cdot \vec{b} - 2\vec{a} \cdot \vec{b} - 4|\vec{b}|^2 = 0$

$10\,\vec{a} \cdot \vec{b} + 6 \cdot 24 - 4 \cdot 36 = 0$

$\vec{a} \cdot \vec{b} = 0$

C. Scalar Product of Vectors in a Rectangular Coordinate System

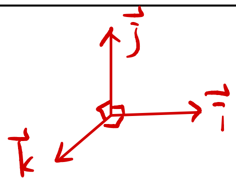
Properties

$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$

(a) $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$

$\vec{i}, \vec{j}, \vec{k} \rightarrow$ orthogonal unit vectors

(b) $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$



Let $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ and θ is the angle between **a** and **b**, where $0° \le \theta \le 180°$, then

$$\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$$

Note that if $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, then $|\mathbf{a}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

Example

10. Let $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

(a) Find $\mathbf{a} \cdot \mathbf{b}$.

(b) Find the angle between **a** and **b**. (Give your answers correct to 0.1°)

(a) $\vec{a} \cdot \vec{b} = 3 \cdot 1 + (-1) \cdot 2 + 2 \cdot 2$

$= 5$

(b) $|\vec{a}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{1^2 + 2^2 + 2^2} = 3$

required angle $= \cos^{-1} \dfrac{5}{3\sqrt{14}}$

$= 63.5°$

Check your concept

Question 1	Question 2
Let $\mathbf{u}=3\mathbf{i}+2\mathbf{j}$ and $\mathbf{v}=\mathbf{i}-9\mathbf{j}$. Find $\mathbf{u}\cdot\mathbf{v}$. $\vec{u}\cdot\vec{v}$ $= 3\cdot 1 + 2\cdot(-9)$ $= -15$	Let $\mathbf{u}=3\mathbf{a}+2\mathbf{b}$ and $\mathbf{v}=\mathbf{a}-9\mathbf{b}$, where $\lvert\mathbf{a}\rvert=2$, $\lvert\mathbf{b}\rvert=5$ and $\mathbf{a}\cdot\mathbf{b}=3$. Find $\mathbf{u}\cdot\mathbf{v}$. $\vec{u}\cdot\vec{v}$ $= (3\vec{a}+2\vec{b})\cdot(\vec{a}-9\vec{b})$ $= 3\lvert\vec{a}\rvert^2 - 25\,\vec{a}\cdot\vec{b} - 18\lvert\vec{b}\rvert^2$ $= 3\cdot 4 - 25\cdot 3 - 18\cdot 25$ $= -513$

Example

11. Let $\mathbf{a}=2\mathbf{i}-\mathbf{j}$, $\mathbf{b}=3\mathbf{i}+k\mathbf{j}$ and $\mathbf{c}=-\mathbf{i}-3\mathbf{j}$.
   (a) Find the value of k when $\mathbf{a}\perp\mathbf{b}$.
   (b) Find the value of $\mathbf{b}\cdot\mathbf{c}$. Hence find the angle between the vectors $\mathbf{b}$ and $\mathbf{c}$ correct to 3 significant figures.

(a) $\vec{a}\cdot\vec{b} = 0$

$2\cdot 3 + (-1)\cdot k = 0$

$k = 6$

(b) $\vec{b} = 3\vec{i} + 6\vec{j}$

$\vec{b}\cdot\vec{c} = 3(-1) + 6(-3) = -21$

$\lvert\vec{b}\rvert = \sqrt{3^2+6^2} = \sqrt{45}$

$\lvert\vec{c}\rvert = \sqrt{(-1)^2+3^2} = \sqrt{10}$

required angle

$= \cos^{-1}\left(\dfrac{-21}{\sqrt{45}\cdot\sqrt{10}}\right)$

$= 172^\circ$

12. **a** and **b** are two vectors such that the angle between **a** and **b** is $\frac{\pi}{6}$. It is given that **b** is a unit vector and $\mathbf{a}\cdot\mathbf{b}=3$.
(a) Find the magnitude of **a**.
(b) Find the magnitude of $2\mathbf{a}+\mathbf{b}$.

(a) $\vec{a}\cdot\vec{b}=3$

$|\vec{a}|\cdot|\vec{b}|\cdot\cos\frac{\pi}{6}=3$

$|\vec{a}|\cdot 1\cdot\frac{\sqrt{3}}{2}=3$

$|\vec{a}|=2\cdot\frac{3}{\sqrt{3}}$

$=2\sqrt{3}$

(b) $|2\vec{a}+\vec{b}|^2$

$=(2\vec{a}+\vec{b})(2\vec{a}+\vec{b})$

$=4|\vec{a}|^2+4\vec{a}\vec{b}+|\vec{b}|^2$

$=4\cdot 12+4\cdot 3+1$

$=61$

$\therefore |2\vec{a}+\vec{b}|=\sqrt{61}$

13. It is given that $|\mathbf{a}|=4$, $|\mathbf{b}|=3$ and the angle between **a** and **b** is 60°. Find the value of k if $\mathbf{a}-2\mathbf{b}$ and $k\mathbf{a}+3\mathbf{b}$ are orthogonal.

$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos 60^\circ=4\cdot 3\cdot\frac{1}{2}=6$

$(\vec{a}-2\vec{b})\cdot(k\vec{a}+3\vec{b})=0$

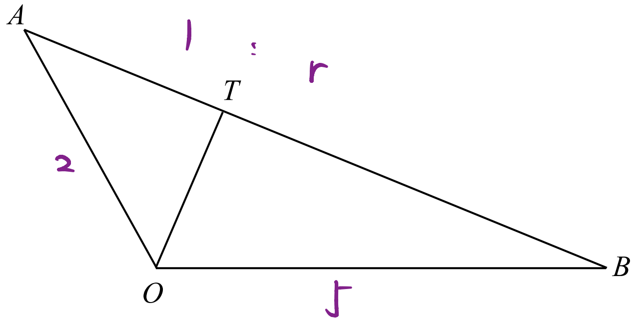
$k|\vec{a}|^2+3\vec{a}\cdot\vec{b}-2k\vec{a}\cdot\vec{b}-6|\vec{b}|^2=0$

$16k+18-12k-54=0$

$k=9$

$|\vec{a}| = 2 \quad |\vec{b}| = 5$

14. In the figure, $OA = 2$, $OB = 5$ and T is a point on AB such that $AT:TB = 1:r$, $\angle AOB = 120°$.
   (a) Express $\mathbf{OT}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and r, where $\mathbf{a} = \mathbf{OA}$ and $\mathbf{b} = \mathbf{OB}$.
   (b) Find $\mathbf{a} \cdot \mathbf{b}$. Hence express $\mathbf{OT}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ if $\angle OTB = 90°$. $\leftarrow \vec{OT} \cdot \vec{AB} = 0$



(a) $\vec{OT} = \dfrac{r \cdot \vec{a} + 1 \cdot \vec{b}}{r+1}$

$= \dfrac{r}{r+1}\vec{a} + \dfrac{1}{r+1}\vec{b}$

(b) $\vec{AB} = \vec{AO} + \vec{OB}$

$= \vec{b} - \vec{a}$

$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos 120°$

$= 2 \cdot 5 \cdot (-\frac{1}{2})$

$= -5$

$\because OT \perp AB$, $\vec{OT} \cdot \vec{AB} = 0$

$\left(\dfrac{r}{r+1}\vec{a} + \dfrac{1}{r+1}\vec{b}\right) \cdot (\vec{b} - \vec{a}) = 0$

$r\,\vec{a} \cdot \vec{b} - r|\vec{a}|^2 + |\vec{b}|^2 - \vec{a} \cdot \vec{b} = 0$

$r(-5) - r \cdot 4 + 25 + 5 = 0$

$r = \dfrac{10}{3}$

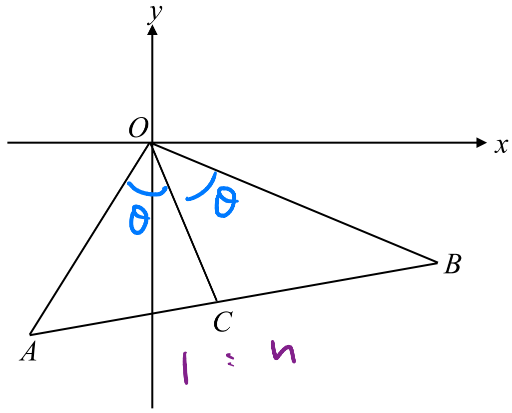
$\vec{OT} = \dfrac{10}{13}\vec{a} + \dfrac{3}{13}\vec{b}$

15. In the figure, $\mathbf{OA} = -\mathbf{i} - 2\mathbf{j}$ and $\mathbf{OB} = 3\mathbf{i} - \mathbf{j}$. C is a point on AB such that $AC : CB = 1 : n$.

(a) Find $\mathbf{OC}$ in terms of n.

(b) Find $\mathbf{OA} \cdot \mathbf{OC}$ and $\mathbf{OB} \cdot \mathbf{OC}$.

Hence find the value of n if OC is the angle bisector of $\angle AOB$.



(a) $\vec{OC} = \dfrac{n \cdot \vec{OA} + 1 \cdot \vec{OB}}{n+1}$

$= \dfrac{-n\vec{i} - 2n\vec{j} + 3\vec{i} - \vec{j}}{n+1}$

$= \dfrac{3-n}{n+1}\vec{i} + \dfrac{-2n-1}{n+1}\vec{j}$

(b) $\vec{OA} \cdot \vec{OC} = \dfrac{n-3}{n+1} + \dfrac{4n+2}{n+1} = \dfrac{5n-1}{n+1}$

$\vec{OB} \cdot \vec{OC} = \dfrac{9-3n}{n+1} + \dfrac{2n+1}{n+1} = \dfrac{10-n}{n+1}$

$\vec{OA} \cdot \vec{OC} = |\vec{OA}| \cdot \boxed{|\vec{OC}| \cdot \cos\theta}$

$\vec{OB} \cdot \vec{OC} = |\vec{OB}| \cdot \boxed{|\vec{OC}| \cdot \cos\theta}$

$|\vec{OA}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|\vec{OB}| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$\Rightarrow \dfrac{\vec{OA} \cdot \vec{OC}}{|\vec{OA}|} = \dfrac{\vec{OB} \cdot \vec{OC}}{|\vec{OB}|}$

$\therefore \dfrac{5n-1}{\sqrt{5}\,\cancel{(n+1)}} = \dfrac{10-n}{\sqrt{10}\,\cancel{(n+1)}}$

$5\sqrt{2}n - \sqrt{2} = 10 - n$

$(5\sqrt{2}+1)n = 10 + \sqrt{2}$

$n = \dfrac{10+\sqrt{2}}{5\sqrt{2}+1} \cdot \dfrac{5\sqrt{2}-1}{5\sqrt{2}-1}$

$= \dfrac{50\sqrt{2} - 10 + 10 - \sqrt{2}}{49}$

$= \dfrac{49\sqrt{2}}{49} = \sqrt{2}$

16. In the figure, BGN and ONA are straight lines such that $BN \perp OA$. It is given that $\angle AOB = 60°$, $ON = \lambda OA$ and $BG : GN = 8 : 1$. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 3$.

(a) (i) Express $\overrightarrow{BN}$ in terms of λ, $\mathbf{a}$ and $\mathbf{b}$.

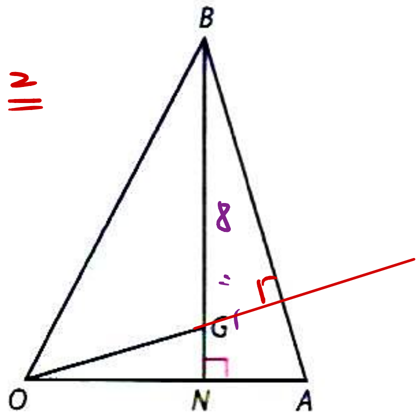
(ii) Hence find the value of λ.

(b) (i) Express $\overrightarrow{OG}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) Prove that G is the orthocentre of ΔAOB.

intersection of 3 altitudes.

$\Rightarrow OG \perp AB$



(a) (i) $\overrightarrow{ON} = \lambda\vec{a}$

$\overrightarrow{BN} = \overrightarrow{BO} + \overrightarrow{ON} = -\vec{b} + \lambda\vec{a}$

(ii) $\overrightarrow{BN} \cdot \overrightarrow{OA} = 0$

$(\lambda\vec{a} - \vec{b}) \cdot \vec{a} = 0$

$\lambda|\vec{a}|^2 - \vec{a} \cdot \vec{b} = 0$

$\lambda \cdot 4 - 2 \cdot 3 \cdot \cos 60° = 0$

$4\lambda - 3 = 0$

$\lambda = \frac{3}{4}$

(b) (i) $\overrightarrow{OG} = \dfrac{8\overrightarrow{ON} + \overrightarrow{OB}}{9} = \dfrac{8 \times \frac{3}{4}\vec{a} + \vec{b}}{9} = \frac{2}{3}\vec{a} + \frac{1}{9}\vec{b}$

(ii) $\overrightarrow{OG} \cdot \overrightarrow{AB} = (\frac{2}{3}\vec{a} + \frac{1}{9}\vec{b}) \cdot (\vec{b} - \vec{a})$

$= \frac{2}{3}\vec{a} \cdot \vec{b} - \frac{2}{3}|\vec{a}|^2 + \frac{1}{9}|\vec{b}|^2 - \frac{1}{9}\vec{a} \cdot \vec{b}$

$= \frac{2}{3} \cdot 3 - \frac{2}{3} \cdot 4 + \frac{1}{9} \cdot 9 - \frac{1}{9} \cdot 3$

$= 0$

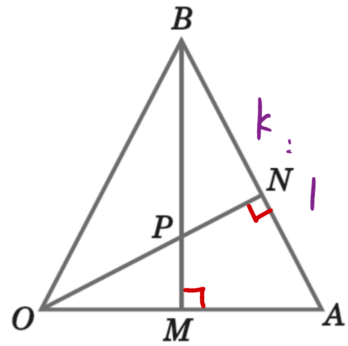
$\therefore OG \perp AB$

$\therefore G$ is the orthocentre of ΔAOB.

17. In the figure, $\overrightarrow{OA} = 4\mathbf{i}$ and $\overrightarrow{OB} = 2\mathbf{i} + 3\mathbf{j}$. M is the mid-point of OA and N lies on AB such that $BN : NA = k : 1$, where $k > 0$. BM intersects ON at P.

(a) Express $\overrightarrow{ON}$ in terms of k.

(b) If A, N, P and M are concyclic, find the value of k.



(a) $\overrightarrow{ON} = \dfrac{k \cdot \overrightarrow{OA} + \overrightarrow{OB}}{k+1}$

$= \dfrac{4k\vec{i} + 2\vec{i} + 3\vec{j}}{k+1}$

$= \dfrac{4k+2}{k+1}\vec{i} + \dfrac{3}{k+1}\vec{j}$

(b) $\overrightarrow{OM} = 2\vec{i}$

$\overrightarrow{BM} = \overrightarrow{BO} + \overrightarrow{OM} = -2\vec{i} - 3\vec{j} + 2\vec{i} = -3\vec{j}$

$\overrightarrow{OA} \cdot \overrightarrow{BM} = 4\vec{i} \cdot (-3\vec{j}) = 0$

$BM \perp OA$

$\because$ A, N, P, M are concyclic, $ON \perp AB$.

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -2\vec{i} + 3\vec{j}$

$\overrightarrow{ON} \cdot \overrightarrow{AB} = 0$

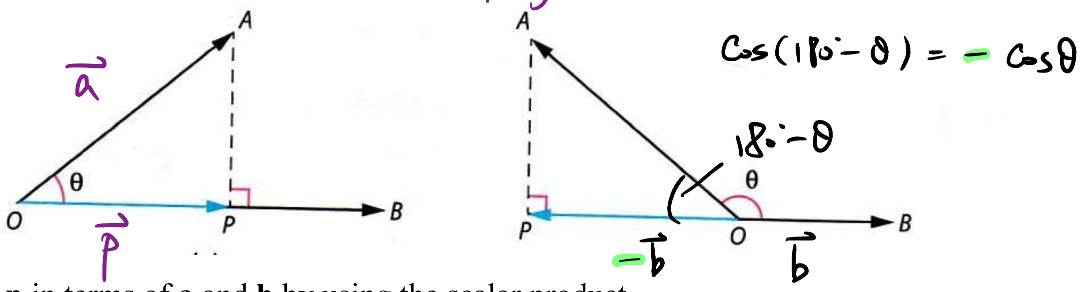
$-2(4k+2) + 3 \cdot 3 = 0$

$-8k - 4 + 9 = 0$

$k = \dfrac{5}{8}$

D. Projection of Vectors

In the following figure, P is a point on OB such that $AP \perp OB$. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OP} = \mathbf{p}$ and θ be the angle between **a** and **b**. **p** is called the projection of **a** onto **b**.



(i) Express **p** in terms of **a** and **b** by using the scalar product.

$|\vec{p}| = |\vec{a}| \cdot \cos\theta$

magnitude

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$

$\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = |\vec{a}|\cdot\cos\theta = |\vec{p}|$

(ii) Find $\mathbf{a}\cdot\hat{\mathbf{b}}$

O, P, B collinear. $\vec{p} = |\vec{p}|\cdot\hat{b}$ ← unit vector in the direction of $\vec{b}$

$\vec{p} = |\vec{p}|\cdot\hat{b}$

$= \dfrac{\vec{a}\odot\vec{b}}{|\vec{b}|} \odot \dfrac{\vec{b}}{|\vec{b}|} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\cdot\vec{b}$

dot product

乘法

magnitude

unit vector

Projection of Vectors

Projection of a non-zero vector **a** onto another non-zero vector **b** $= \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$.

Magnitude of the projection of **a** onto **b** $= \dfrac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|}$

Example

18. Find the projection of $4\mathbf{i}+7\mathbf{j}$ onto $2\mathbf{i}-3\mathbf{j}$.

$$\text{projection vector} = \frac{(4\vec{i}+7\vec{j})\cdot(2\vec{i}-3\vec{j})}{\left(\sqrt{2^2+(-3)^2}\right)^2}(2\vec{i}-3\vec{j})$$

$$= \frac{4\cdot 2+7\cdot(-3)}{13}(2\vec{i}-3\vec{j})$$

$$= -(2\vec{i}-3\vec{j})$$

$$= -2\vec{i}+3\vec{j}$$

19. Find the projection of $2\mathbf{i}+5\mathbf{j}$ onto $3\mathbf{i}-2\mathbf{j}$.

$$\text{projection vector} = \frac{(2\vec{i}+5\vec{j})(3\vec{i}-2\vec{j})}{3^2+2^2}(3\vec{i}-2\vec{j})$$

$$= \frac{6-10}{13}(3\vec{i}-2\vec{j})$$

$$= \frac{-12}{13}\vec{i}+\frac{8}{13}\vec{j}$$

20. It is given that $\mathbf{u} = -2\mathbf{i} + \lambda\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. If the projection of $\mathbf{u}$ onto $\mathbf{v}$ is $\mu\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{4}{3}\mathbf{k}$, find the values of λ and μ.

projection of $\vec{u}$ onto $\vec{v}$

$$= \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v}$$

$$= \frac{-2-2\lambda+2}{1^2+2^2+2^2} \cdot (\vec{i} - 2\vec{j} + 2\vec{k})$$

$$= \frac{-2\lambda}{9} (\vec{i} - 2\vec{j} + 2\vec{k})$$

$$= \frac{-2\lambda}{9}\vec{i} + \frac{4\lambda}{9}\vec{j} - \frac{4\lambda}{9}\vec{k}$$

$$\frac{4\lambda}{9} = \frac{4}{3} \qquad \frac{-2\lambda}{9} = \mu$$

$$\lambda = 3 \qquad \mu = \frac{-2}{3}$$

21. It is given that $\mathbf{a}=\mathbf{i}+2\mathbf{j}-5\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.
   (a) Find the projection of **a** onto **b**.
   (b) Find the magnitude of the projection of $\mathbf{a}+\mathbf{b}$ onto **a**.

(a) projection of $\vec{a}$ onto $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\cdot\vec{b}$$

$$= \frac{2-4-15}{2^2+2^2+3^2}\cdot(2\vec{i}-2\vec{j}+3\vec{k})$$

$$= -(2\vec{i}-2\vec{j}+3\vec{k})$$

$$= -2\vec{i}+2\vec{j}-3\vec{k}$$

(b) $\vec{a}+\vec{b} = 3\vec{i}-2\vec{k}$

required magnitude

$$= \frac{(\vec{a}+\vec{b})\cdot\vec{a}}{|\vec{a}|}$$

$$= \frac{3+(-2)(-5)}{\sqrt{1^2+2^2+5^2}}$$

$$= \frac{13}{\sqrt{30}}$$

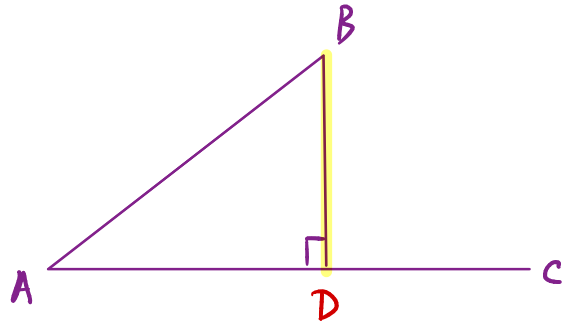
$$= \frac{13\sqrt{30}}{30}$$

22. Let $\overrightarrow{AB} = 3\mathbf{i} - 3\mathbf{j}$ and $\overrightarrow{AC} = -4\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$.

$\overrightarrow{AD}$

(a) Find the projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$.

(b) Find the distance between B and AC.



(a) Let D be the projection of B on AC.

$$\overrightarrow{AD} = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AC}|^2}\cdot\overrightarrow{AC}$$

$$= \frac{3\cdot(-4) + (-3)\cdot 5}{4^2+5^2+2^2}\cdot(-4\vec{i} + 5\vec{j} - 2\vec{k})$$

$$= \frac{-3}{5}(-4\vec{i} + 5\vec{j} - 2\vec{k})$$

$$= \frac{12}{5}\vec{i} - 3\vec{j} + \frac{6}{5}\vec{k}$$

$$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -3\vec{i} + 3\vec{j} + \frac{12}{5}\vec{i} - 3\vec{j} + \frac{6}{5}\vec{k}$$

$$= -\frac{3}{5}\vec{i} + \frac{6}{5}\vec{k}$$

$\therefore$ required shortest distance

$$= \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{6}{5}\right)^2}$$

$$= \sqrt{\frac{9}{5}}$$

$$= \frac{3\sqrt{5}}{5}$$

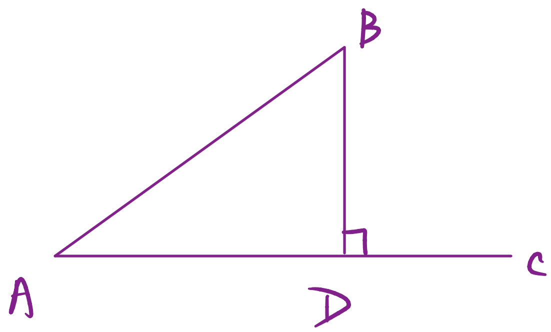
23. Let $A(2, -2)$, $B(3, 0)$ and $C(4, -1)$ be three points on a 2-dimensional coordinate plane.

(a) Find the projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$.

(b) Find the distance between B to AC.

(a) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA} = \underbrace{\vec{i}}_{3-2} + \underbrace{2\vec{j}}_{0-(-2)}$

$\overrightarrow{AC} = 2\vec{i} + \vec{j}$



Let D be the projection of B on AC.

$$\overrightarrow{AD} = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AC}|^2} \cdot \overrightarrow{AC}$$

$$= \frac{1 \cdot 2 + 2 \cdot 1}{2^2 + 1^2} \cdot (2\vec{i} + \vec{j})$$

$$= \frac{4}{5}(2\vec{i} + \vec{j})$$

$$= \frac{8}{5}\vec{i} + \frac{4}{5}\vec{j}$$

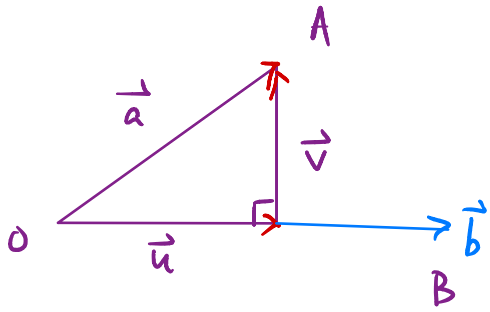
(b) $|\overrightarrow{AB}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|\overrightarrow{AD}| = \sqrt{\left(\frac{8}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{4}{\sqrt{5}}$

$\therefore$ required shortest distance

$= |\overrightarrow{BD}| = \sqrt{5 - \frac{16}{5}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$

24. $A(2, 5, -2)$ and $B(2, -1, 1)$ are two points in a 3-dimensional coordinate system. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\mathbf{a} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u}$ is parallel to $\mathbf{b}$ and $\mathbf{v}$ is perpendicular to $\mathbf{b}$.
    (a) Find $\mathbf{u}$ and $\mathbf{v}$.
    (b) Hence, find the distance between A and OB.



(a) $\vec{u}$ = projection of $\vec{a}$ onto $\vec{b}$

$\vec{v} = -\vec{u} + \vec{a}$

$$\vec{u} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\cdot|\vec{b}|$$

$$= \frac{(2\vec{i}+5\vec{j}-2\vec{k})\cdot(2\vec{i}-\vec{j}+\vec{k})}{2^2+1^2+1^2}\cdot(2\vec{i}-\vec{j}+\vec{k})$$

$$= \frac{2\cdot 2+5(-1)-2\cdot 1}{6}(2\vec{i}-\vec{j}+\vec{k})$$

$$= -\frac{1}{2}(2\vec{i}-\vec{j}+\vec{k})$$

$$= -\vec{i}+\frac{1}{2}\vec{j}-\frac{1}{2}\vec{k}$$

$$\vec{v} = -\vec{u}+\vec{a} = \vec{i}-\frac{1}{2}\vec{j}+\frac{1}{2}\vec{k}+2\vec{i}+5\vec{j}-2\vec{k}$$

$$= 3\vec{i}+\frac{9}{2}\vec{j}-\frac{3}{2}\vec{k}$$

(b) required distance $= |\vec{v}|$

$$= \sqrt{3^2+\left(\frac{9}{2}\right)^2+\left(\frac{3}{2}\right)^2}$$

$$= \frac{3\sqrt{7}}{\sqrt{2}}$$

$$= \frac{3\sqrt{14}}{2}$$

result is a vector

13.3 Vector Products

Cross Product

A. Definition of Vector Product

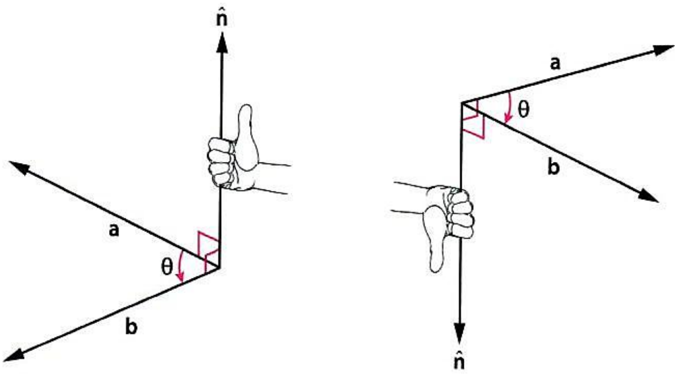
(a) The vector product of vectors **a** and **b**, denoted by $\mathbf{a}\times\mathbf{b}$, is defined as

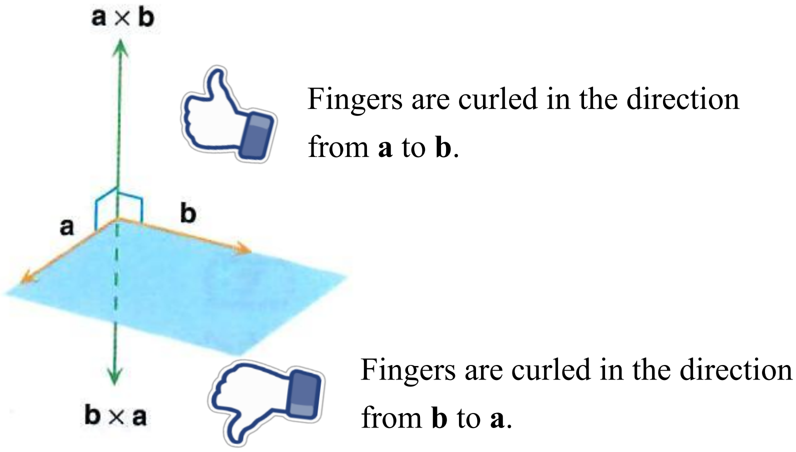
magnitude

$$\mathbf{a}\times\mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin\theta)\hat{\mathbf{n}},$$ ← unit vector

where θ is the angle between **a** and **b**, and $0° \le \theta \le 180°$,

$\hat{\mathbf{n}}$ is the unit vector perpendicular to both **a** and **b**, and its direction is determined by the **right-hand rule.**





(b) Note that $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.

(c) $\mathbf{a}\times\mathbf{b}$ is a ___vector___ but not a scalar.

(d) If **a** and **b** is a zero vector, then $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.

(e) For non-zero vectors **a** and **b**, $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ if and only if **a** is parallel to **b**.

Example

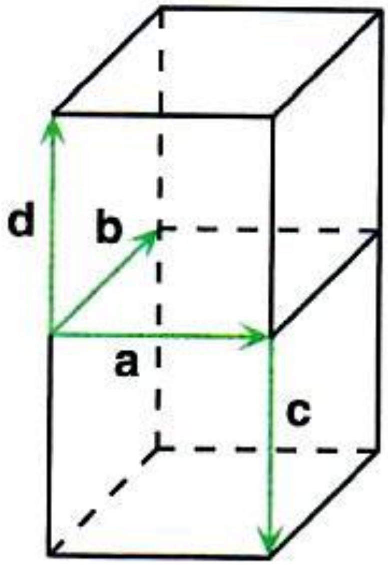
25. In the figure, the solid is formed by joining two unit cubes. Find the following vector products.

(a) $\mathbf{a}\times\mathbf{b} =$ $\vec{d}$

(b) $\mathbf{a}\times\mathbf{c} =$ $\vec{b}$

(c) $\mathbf{b}\times\mathbf{d} =$ $\vec{a}$

(d) $\mathbf{b}\times\mathbf{a} =$ $\vec{c} = -\vec{d}$



26. In each of the following, θ is the angle between **a** and **b**. Find $|\mathbf{a}\times\mathbf{b}|$.

(a) $|\mathbf{a}|=1$, $|\mathbf{b}|=2$, $\theta=45°$

(b) $|\mathbf{a}|=9$, $|\mathbf{b}|=6$, $\theta=30°$

(a) $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\cdot\sin 45° = 1\cdot 2\cdot\frac{\sqrt{2}}{2} = \sqrt{2}$

(b) $|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin 30° = 9\cdot 6\cdot\frac{1}{2} = 27$

B. Properties of Vector Products

For any vectors **a**, **b** and **c**, we have

(a) $\mathbf{a}\times\mathbf{a}=\mathbf{0}$

(b) $\mathbf{a}\times\mathbf{0}=\mathbf{0}$

(c) $\mathbf{b}\times\mathbf{a}=-(\mathbf{a}\times\mathbf{b})$

(d) $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$

(e) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$

(f) $(\lambda\mathbf{a})\times\mathbf{b}=\lambda(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(\lambda\mathbf{b})$

(g) $|\mathbf{a}\times\mathbf{b}|=|\mathbf{b}\times\mathbf{a}|$

(h) $|\mathbf{a}\times\mathbf{b}|^2=|\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2$ ← connect dot product and cross product

Note that

$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$

but

$\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$

Example

27. If $|\mathbf{a}\times\mathbf{b}|=\frac{1}{2}$, find $|(2\mathbf{a}+\mathbf{b})\times(\mathbf{a}-2\mathbf{b})|$.

$|(2\vec{a}+\vec{b})\times(\vec{a}-2\vec{b})|$

$=|2\vec{a}\times\vec{a}-4(\vec{a}\times\vec{b})+\vec{b}\times\vec{a}-2\vec{b}\times\vec{b}|$

$=|-5(\vec{a}\times\vec{b})|$ $\quad | \ | \leftarrow$ absolute value

$=+\frac{5}{2}$

28. It is given that $|\mathbf{a}|=3$, $|\mathbf{b}|=2$ and the angle between **a** and **b** is $45°$.

(a) Prove that $(\mathbf{a}+\mathbf{b})\times(3\mathbf{b}-\mathbf{a})=4(\mathbf{a}\times\mathbf{b})$.

(b) Find $|(\mathbf{a}+\mathbf{b})\times(3\mathbf{b}-\mathbf{a})|$.

(a) $(\vec{a}+\vec{b})\times(3\vec{b}-\vec{a})$

$=3(\vec{a}\times\vec{b})-\vec{a}\times\vec{a}+3\vec{b}\times\vec{b}-\vec{b}\times\vec{a}$

$=3(\vec{a}\times\vec{b})+\vec{a}\times\vec{b}$

$=4(\vec{a}\times\vec{b})$

(b) $|4(\vec{a}\times\vec{b})|$

$=4|\vec{a}|\cdot|\vec{b}|\cdot\sin 45°$

$=4\cdot3\cdot2\cdot\frac{\sqrt{2}}{2}$

$=12\sqrt{2}$

$$|\vec{a} \times \vec{b}|^2$$

$$= (|\vec{a}||\vec{b}| \sin\theta)^2$$

$$= |\vec{a}|^2 \cdot |\vec{b}|^2 \cdot \sin^2\theta$$

$$= |\vec{a}|^2 \cdot |\vec{b}|^2 \cdot (1 - \cos^2\theta)$$

$$= |\vec{a}|^2 \cdot |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$

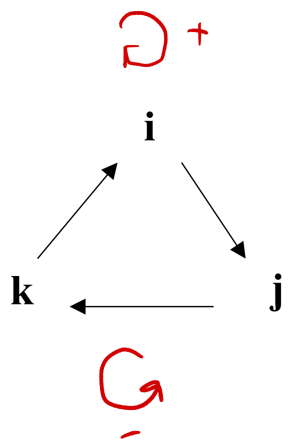
$$= |\vec{a}|^2 \cdot |\vec{b}|^2 - (|\vec{a}||\vec{b}| \cdot \cos\theta)^2$$

$$= |\vec{a}|^2 \cdot |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

C. Vector Product of Vectors in $\mathbf{R}^3$

By the definition of vector product, we have

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$$



$\mathbf{i}\times\mathbf{j}=$ $\vec{k}$　　$\mathbf{i}\cdot\mathbf{j}=$ 0

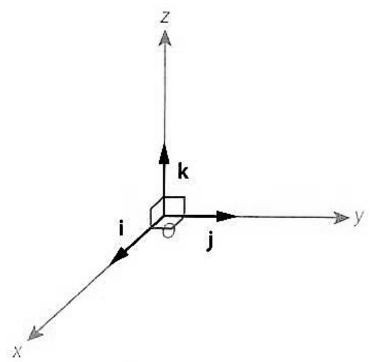
$\mathbf{j}\times\mathbf{k}=$ $\vec{i}$　　$\mathbf{j}\cdot\mathbf{k}=$ 0

$\mathbf{k}\times\mathbf{i}=$ $\vec{j}$　　$\mathbf{k}\cdot\mathbf{i}=$ 0

$\mathbf{j}\times\mathbf{i}=-\mathbf{k}$

$\mathbf{k}\times\mathbf{j}=$ $-\vec{i}$

$\mathbf{i}\times\mathbf{k}=$ $-\vec{j}$



Example

29. Find the following vector products.

(a) $\mathbf{k}\times(2\mathbf{i}-\mathbf{j})$

$= 2\vec{j} - (-\vec{i})$

$= \vec{i} + 2\vec{j}$

(b) $(4\mathbf{i}-\mathbf{j}-2\mathbf{k})\times(\mathbf{i}+\mathbf{j})$

$= 4\vec{k} - (-\vec{k}) - 2\vec{j} - 2(-\vec{i})$

$= 2\vec{i} - 2\vec{j} + 5\vec{k}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & -2 \\ 1 & 1 & 0 \end{vmatrix}$$

$= -2\vec{j} + 4\vec{k} - (-\vec{k} - 2\vec{i})$

$= 2\vec{i} - 2\vec{j} + 5\vec{k}$

In fact, it is more effective to find the vector product $\mathbf{a}\times\mathbf{b}$ using a determinant.

Given that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

* order

Example

30. It is given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.
    (a) Find $\mathbf{a}\times\mathbf{b}$.
    (b) Find $\mathbf{b}\times\mathbf{a}$.

(a) $\vec{a}\times\vec{b}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix}$$

$= 6\vec{i} - \vec{j} - 4\vec{k} - (3\vec{k} + 2\vec{i} + 4\vec{j})$

$= 4\vec{i} - 5\vec{j} - 7\vec{k}$

(b) $\vec{b}\times\vec{a}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$

$= 2\vec{i} + 4\vec{j} + 3\vec{k} - (6\vec{i} - \vec{j} - 4\vec{k})$

$= -4\vec{i} + 5\vec{j} + 7\vec{k}$

31. It is given that $\mathbf{a} = \mathbf{i} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
    (a) Find $\mathbf{a}\times\mathbf{b}$.
    (b) Find $\mathbf{a}\cdot\mathbf{b}$.

(a) $\vec{a}\times\vec{b}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 3 \\ 1 & 1 & -2 \end{vmatrix}$$

$= 3\vec{j} + \vec{k} - (3\vec{i} - 2\vec{j})$

$= -3\vec{i} + 5\vec{j}$

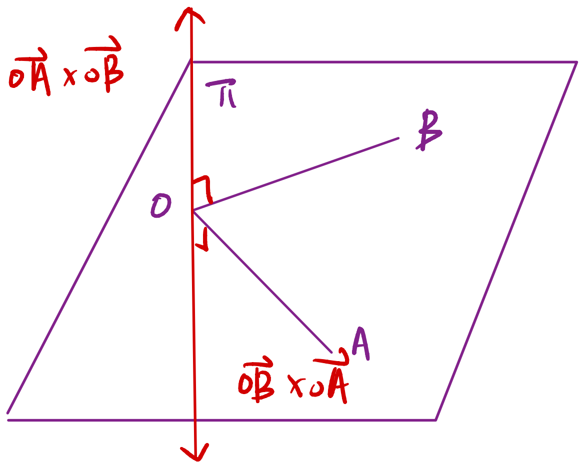
D. Applications of Vector Products

I. Vectors Orthogonal to Two Given Vectors

Example

32. Let $\overrightarrow{OA} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OB} = -4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Denote the plane containing O, A and B by Π.

    Find the two unit vectors that perpendicular to Π.



$$\vec{OA} \times \vec{OB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -1 \\ -4 & 2 & 3 \end{vmatrix}$$

$$= -6\vec{i} + 4\vec{j} + 4\vec{k} - (8\vec{k} - 2\vec{i} + 6\vec{j})$$

$$= -4\vec{i} - 2\vec{j} - 4\vec{k}$$

$$|\vec{OA} \times \vec{OB}| = \sqrt{4^2 + 2^2 + 4^2} = 6$$

required two unit vectors $= \pm \frac{1}{6}(\vec{OA} \times \vec{OB})$

$$= \frac{-2}{3}\vec{i} - \frac{1}{3}\vec{j} - \frac{4}{3}\vec{k} \text{ and } \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{4}{3}\vec{k}$$

33. Let $\overrightarrow{OP} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OQ} = 4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$. Denote the plane containing O, P and Q by Π.

    Find the two unit vectors that perpendicular to Π.

$$\vec{OP} \times \vec{OQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & -1 \\ 4 & -3 & 3 \end{vmatrix} = 6\vec{i} - 4\vec{j} + 6\vec{k} - (3\vec{i} - 6\vec{j} + 8\vec{k})$$

$$= 3\vec{i} + 2\vec{j} - 2\vec{k}$$

$$|\vec{OP} \times \vec{OQ}| = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$$

$\therefore$ the two unit vectors are

$$\pm \frac{1}{\sqrt{17}}(3\vec{i} + 2\vec{j} - 2\vec{k})$$

$$= \pm \frac{\sqrt{17}}{17}(3\vec{i} + 2\vec{j} - 2\vec{k})$$

34. Let $\overrightarrow{OP} = 2\mathbf{i} + \mathbf{k}$, $\overrightarrow{OQ} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\overrightarrow{OR} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Denote the plane containing P, Q and R by Π.

(a) Find $\overrightarrow{PR} \times \overrightarrow{QR}$.

(b) It is given that $\overrightarrow{OS} = -\mathbf{i} + \mathbf{j}$. Using the result of (a), find the angle between RS and Π.

Hint of (b)

1. Find $\cancel{\overrightarrow{OS}}$ $\overrightarrow{RS}$ $\cdot (\overrightarrow{PR} \times \overrightarrow{QR})$.

2. - If the dot product is positive, the included angle (ϕ) is **acute**.
   - The required angle = $90° - \phi$

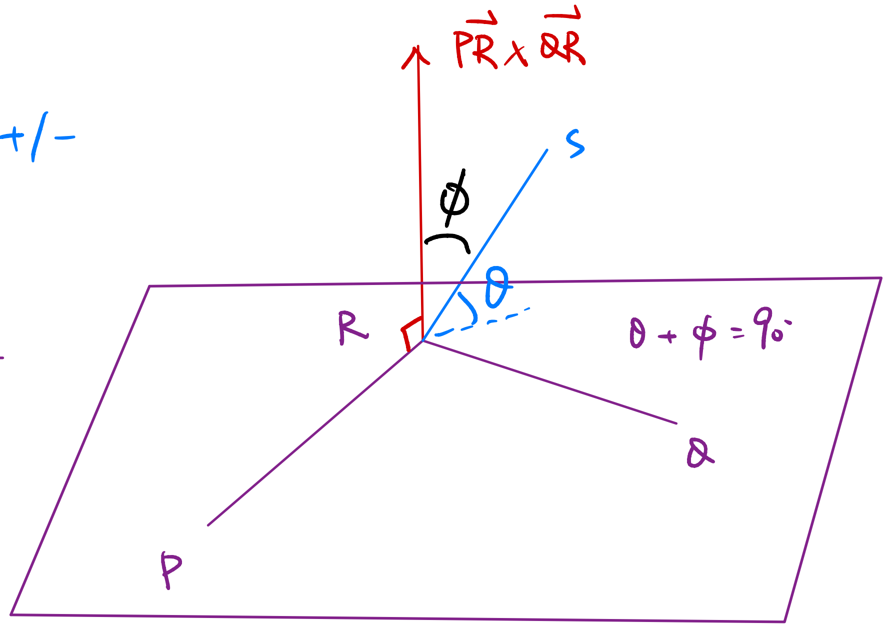
3. - If the dot product is negative, the included angle is **obtuse**.
   - The required angle = $\phi - 90°$

$(\overrightarrow{PR} \times \overrightarrow{QR}) \cdot \overrightarrow{RS}$

(+) (+) +/-

$= |\overrightarrow{PR} \times \overrightarrow{QR}| \cdot |\overrightarrow{RS}| \cdot \cos\phi$

$$\cos\phi = \frac{(\overrightarrow{PR} \times \overrightarrow{QR}) \cdot \overrightarrow{RS}}{|\overrightarrow{PR} \times \overrightarrow{QR}| \cdot |\overrightarrow{RS}|}$$



(a) $\overrightarrow{PR} = -3\vec{i} + 2\vec{j} + 3\vec{k}$

$\overrightarrow{QR} = -2\vec{i} + 3\vec{j} - \vec{k}$

$$\overrightarrow{PR} \times \overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 2 & 3 \\ -2 & 3 & -1 \end{vmatrix}$$

$$= -2\vec{i} - 6\vec{j} - 9\vec{k} - (9\vec{i} + 3\vec{j} - 4\vec{k})$$

$$= -11\vec{i} - 9\vec{j} - 5\vec{k}$$

(b) $\overrightarrow{RS} = -\vec{j} - 4\vec{k}$

Let ϕ be the angle between $\overrightarrow{RS}$ and $\overrightarrow{PR} \times \overrightarrow{QR}$

$(\overrightarrow{PR} \times \overrightarrow{QR}) \cdot \overrightarrow{RS} = -9(-1) - 5(-4) = 29 , > 0$

$\therefore$ ϕ is acute.

$$\cos\phi = \frac{(\overrightarrow{PR} \times \overrightarrow{QR}) \cdot \overrightarrow{RS}}{|\overrightarrow{PR} \times \overrightarrow{QR}|\,|\overrightarrow{RS}|} = \frac{29}{\sqrt{11^2+9^2+5^2} \cdot \sqrt{1^2+4^2}}$$

$$= \frac{29}{\sqrt{3859}}$$

$\therefore$ required angle $= 90^\circ - \phi$

$= 27.8^\circ$

35. Let $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = 3\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OC} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$. Denote the plane containing A, B and C by Π.

(a) Find $\overrightarrow{AC} \times \overrightarrow{BC}$.

(b) It is given that $\overrightarrow{OD} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Using the result of (a), find the angle between CD and Π, correct to the nearest $0.1°$.

Hint of (b)

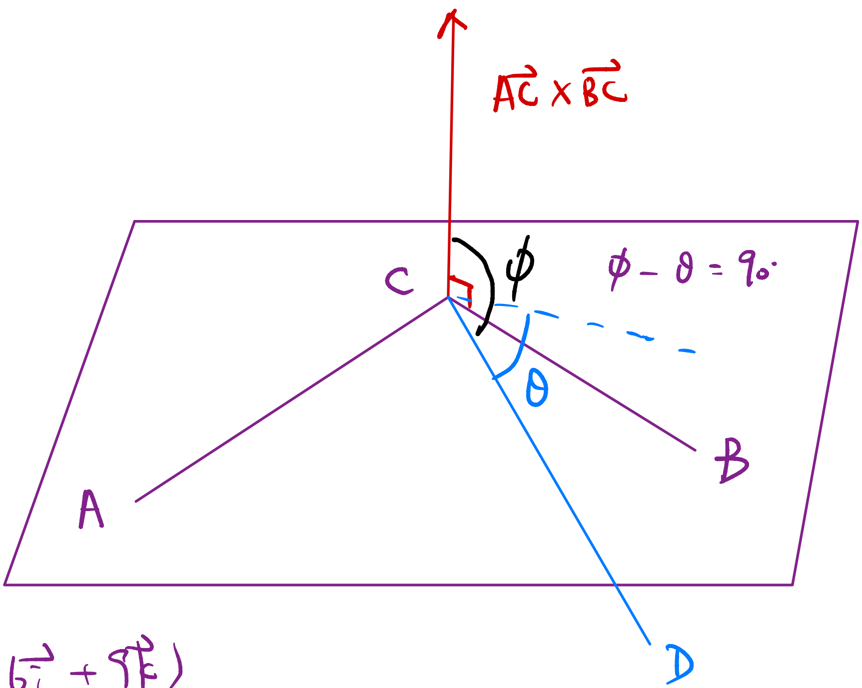
1. Find $\overrightarrow{CD} \cdot (\overrightarrow{AC} \times \overrightarrow{BC})$.

2. - If the dot product is positive, the included angle (ϕ) is **acute**.
   - The required angle = $90° - \phi$

3. - If the dot product is negative, the included angle is **obtuse**.
   - The required angle = $\phi - 90°$

(a) $\vec{AC} = 2\vec{i} + 3\vec{j} - \vec{k}$

$\vec{BC} = 3\vec{i} - 2\vec{j} + 3\vec{k}$



$\vec{AC} \times \vec{BC}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 3 & -2 & 3 \end{vmatrix}$

$= 9\vec{i} - 3\vec{j} - 4\vec{k} - (2\vec{i} + 6\vec{j} + 9\vec{k})$

$= 7\vec{i} - 9\vec{j} - 13\vec{k}$

(b) $\vec{CD} = -\sqrt{5}\vec{i} + \vec{j}$

$\vec{AC} \times \vec{BC} = 7\vec{i} - 9\vec{j} - 13\vec{k}$

$(\vec{AC} \times \vec{BC}) \cdot \vec{CD} = -35 - 9 = -44 < 0$

Let ϕ be the angle between $\vec{CD}$ and $\vec{AC} \times \vec{BC}$

ϕ is obtuse.

$$\cos\phi = \frac{(\vec{AC} \times \vec{BC}) \cdot \vec{CD}}{|\vec{AC} \times \vec{BC}| \cdot |\vec{CD}|}$$

$$= \frac{-44}{\sqrt{7^2+9^2+13^2}\sqrt{5^2+1^2}}$$

$$= \frac{-44}{\sqrt{299} \cdot \sqrt{26}}$$

$\phi = 119.9°$

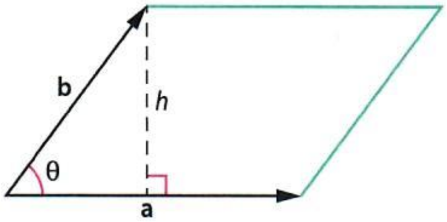
$\therefore$ angle between CD and Π

$= \phi - 90°$

$= 29.9°$

II. Areas of Parallelograms and Triangles

Consider the parallelogram formed by **a** and **b** in the figure.



$h = |\vec{b}| \cdot \sin\theta$

area of //gram

$= |\vec{a}| \cdot h$

$= |\vec{a}| \cdot |\vec{b}| \cdot \sin\theta$

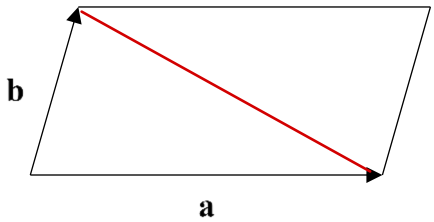
$= |\vec{a} \times \vec{b}|.$

Area of Parallelogram = $|\mathbf{a} \times \mathbf{b}|$

Area of Triangle = $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

$|\quad|$ absolute value

$\Rightarrow$ positive



Example

$|\vec{AB} \times \vec{AC}|$

36. Three points $A(-5, 4, 3)$, $B(-5, 5, 4)$ and $C(-1, 9, 6)$ are given. Find the area of ΔABC.

$\vec{AB} = \vec{j} + \vec{k}$

$\vec{AC} = 4\vec{i} + 5\vec{j} + 3\vec{k}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 4 & 5 & 3 \end{vmatrix}$$

$= 3\vec{i} + 4\vec{j} - (4\vec{k} + 5\vec{i})$

$= -2\vec{i} + 4\vec{j} - 4\vec{k}$

$\therefore$ area of ΔABC

$= \frac{1}{2} \cdot \sqrt{(-2)^2 + 4^2 + (-4)^2}$

$= 3$

37. Four points $P(2, 0, 1)$, $Q(2, 1, 2)$, $R(6, 5, 4)$ and $S(6, 4, 3)$ are given.
   (a) Show that $PQRS$ is a parallelogram.
   (b) Find the area of $PQRS$.

(a) $\vec{PQ} = \vec{j} + \vec{k}$

$\vec{QR} = 4\vec{i} + 4\vec{j} + 2\vec{k}$

$\vec{RS} = -\vec{j} - \vec{k}$

$\vec{PS} = 4\vec{i} + 4\vec{j} + 2\vec{k}$

$\because$ $\vec{RS} = -\vec{PQ}$, $RS // PQ$

$\vec{QR} = \vec{PS}$, $QR // PS$

$\therefore$ $PQRS$ is a // gram

(b) $\vec{PQ} \times \vec{QR}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 4 & 4 & 2 \end{vmatrix}$$

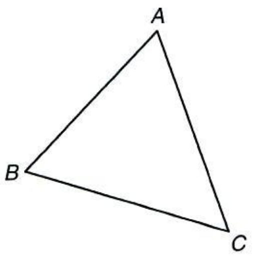
$= 2\vec{i} + 4\vec{j} - 4\vec{k} - 4\vec{i}$

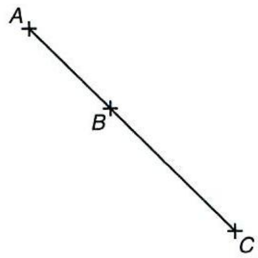
$= -2\vec{i} + 4\vec{j} - 4\vec{k}$

$\therefore$ area of $PQRS$

$= |\vec{PQ} \times \vec{RS}| = \sqrt{(-2)^2 + 4^2 + (-4)^2} = 6$

Suppose A, B and C are three distinct points in a coordinate system.
When area of $\Delta ABC = 0$, the three points A, B and C must lie on the same straight line as shown.





Area of $\Delta ABC = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| \neq 0$ Area of $\Delta ABC = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = 0$

$\vec{AB} \times \vec{AC} = \mathbf{0}$ if and only if A, B and C are collinear.

Example

38. Show that $A(1, 3, -2)$, $B(3, 5, 0)$ and $C(4, 6, 1)$ are collinear by using vector product.

$\vec{AB} = 2\vec{i} + 2\vec{j} + 2\vec{k}$

$\vec{AC} = 3\vec{i} + 3\vec{j} + 3\vec{k}$

$\vec{AB} \times \vec{AC}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix}$$

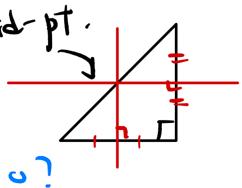
$= \vec{0}$

$\therefore$ area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}| = 0$

$\therefore$ A, B, C are collinear.

39. Let $\overrightarrow{OA} = \mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = 4\mathbf{i} + 2\mathbf{k}$, where O is the origin. It is given that Q is the foot of perpendicular from C to BA produced.

mid-pt.



(a) Find the area of ΔABC.

right-angled Δ

(b) Find $|\overrightarrow{CQ}|$.

$\overrightarrow{QG} \cdot \overrightarrow{BC} = 0$?

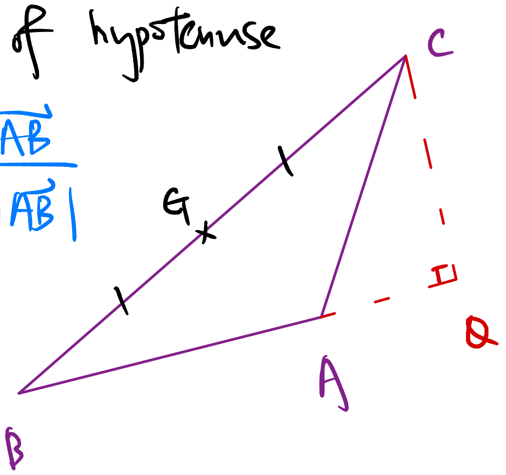
(c) Denote the circumcentre of ΔBCQ by G. Is QG perpendicular to BC? Explain your answer.

↳ mid-pt of hypotenuse

(a) $\overrightarrow{AB} = 2\vec{j} - \vec{k}$

$\overrightarrow{AC} = 4\vec{i} - \vec{j}$

$\overrightarrow{QB} = |\overrightarrow{QB}| \cdot \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$



$\overrightarrow{AB} \times \overrightarrow{AC}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -1 \\ 4 & -1 & 0 \end{vmatrix}$$

$= -4\vec{j} - \vec{i} - 8\vec{k}$

$= -\vec{i} - 4\vec{j} - 8\vec{k}$

area of ΔABC

$= \frac{1}{2}\sqrt{1^2 + 4^2 + 8^2}$

$= \frac{9}{2}$

(b) $|\overrightarrow{AB}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\dfrac{\sqrt{5} \cdot |\overrightarrow{CQ}|}{2} = \dfrac{9}{2}$

$|\overrightarrow{CQ}| = \dfrac{9}{\sqrt{5}} = \dfrac{9\sqrt{5}}{5}$

(c) $\overrightarrow{BG} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(4\vec{i} - 3\vec{j} + \vec{k})$

$= 2\vec{i} - \frac{3}{2}\vec{j} + \frac{1}{2}\vec{k}$

$|\overrightarrow{BC}| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$

$|\overrightarrow{QB}| = \sqrt{|\overrightarrow{BC}|^2 - |\overrightarrow{CQ}|^2} = \sqrt{26 - \frac{81}{5}} = \dfrac{7\sqrt{5}}{5}$

$$\therefore \overrightarrow{QB} = |\overrightarrow{QB}| \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$$

$$= \frac{7\sqrt{5}}{5} \cdot \frac{(2\vec{j} - \vec{k})}{\sqrt{2^2 + 1^2}}$$

$$= \frac{14}{5}\vec{j} - \frac{7}{5}\vec{k}$$

$$\overrightarrow{QG} = \overrightarrow{QB} + \overrightarrow{BG}$$

$$= \frac{14}{5}\vec{j} - \frac{7}{5}\vec{k} + 2\vec{i} - \frac{3}{2}\vec{j} + \frac{1}{2}\vec{k}$$

$$= 2\vec{i} + \frac{13}{10}\vec{j} - \frac{9}{10}\vec{k}$$

$$\overrightarrow{QG} \cdot \overrightarrow{BC} = \left(2\vec{i} + \frac{13}{10}\vec{j} - \frac{9}{10}\vec{k}\right) \cdot \left(4\vec{i} - 3\vec{j} + \vec{k}\right)$$

$$= 2 \cdot 4 + \frac{13}{10} \cdot (-3) - \frac{9}{10} \cdot 1$$

$$= \frac{16}{5}, \quad \neq 0$$

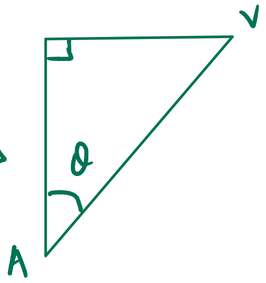
$\therefore$ QG is not perpendicular to BC.

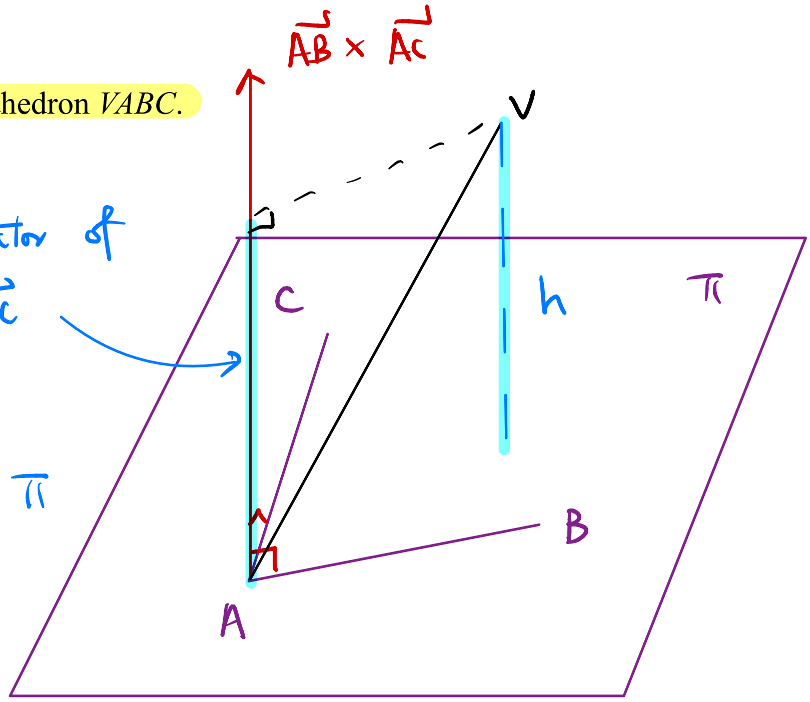
40. The position vectors of A, B, C and V are $-\mathbf{i}+3\mathbf{j}+5\mathbf{k}$, $3\mathbf{i}+\mathbf{j}+\mathbf{k}$, $11\mathbf{i}+7\mathbf{j}+3\mathbf{k}$ and $5\mathbf{i}+7\mathbf{j}+15\mathbf{k}$ respectively. Denote the plane which contains A, B and C by Π.

(a) Find $\overrightarrow{AB}\times\overrightarrow{AC}$.

(b) Find the distance from V to Π.

(c) Hence, find the volume of the tetrahedron $VABC$.





projection vector of $\vec{AV}$ onto $\vec{AB}\times\vec{AC}$

↑

magnitude = distance from V to π

$$= \frac{\vec{AV}\cdot(\vec{AB}\times\vec{AC})}{|\vec{AB}\times\vec{AC}|}$$

= height of VABC correspond to Δ ABC.

$$\frac{|\vec{AV}|\cdot\cancel{|\vec{AB}\times\vec{AC}|}\cdot\cos\theta}{\cancel{|\vec{AB}\times\vec{AC}|}}$$

$= |\vec{AV}|\cdot\cos\theta$

volume of VABC

$= \frac{1}{3}\cdot$ area of ΔABC × h

(a) $\vec{AB} = 4\vec{i} - 2\vec{j} - 4\vec{k}$

$\vec{AC} = 12\vec{i} + 4\vec{j} - 2\vec{k}$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & -4 \\ 12 & 4 & -2 \end{vmatrix}$$

$= 4\vec{i} - 48\vec{j} + 16\vec{k} - (-24\vec{k} - 16\vec{i} - 8\vec{j})$

$= 20\vec{i} - 40\vec{j} + 40\vec{k}$

(b) $\overrightarrow{AV} = 6\vec{i} + 4\vec{j} + 10\vec{k}$

distance from V to Π

= magnitude of projection vector of $\overrightarrow{AV}$ onto $(\overrightarrow{AB} \times \overrightarrow{AC})$

$$= \frac{\overrightarrow{AV} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$$

$$= \frac{6 \cdot 20 + 4(-40) + 40 \cdot 10}{\sqrt{20^2 + 40^2 + 40^2}}$$

$$= \frac{360}{60}$$

$= 6$

(c) distance from V to Π

= height of tetrahedron VAB correspond to base ΔABC

area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = 30$

volume of VABC

$= \frac{1}{3} \cdot 30 \cdot 6$

$= 60$