Chapter 9 Applications of Definite Integration
Supplementary Notes

Name: ______________________________ (   )   Class: F.5 _____

9.1 Areas of Plane Figures

A. Area between a Curve and the x-axis

Let $y = f(x)$ be a continuous function defined on the interval $[a, b]$.

If $f(x) \geq 0$ for all x in the interval $[a, b]$, then the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ can be found by integrating $f(x)$ with respect to x from a to b,

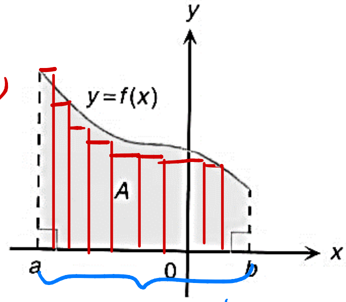
i.e. Area $A = \int_a^b f(x)\mathrm{d}x$

$\int_a^b y\,dx$

sum $n \to \infty$

$y = f(x)$

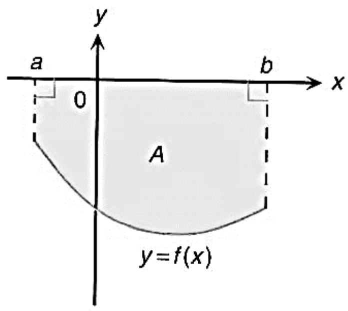
dx



n scripts

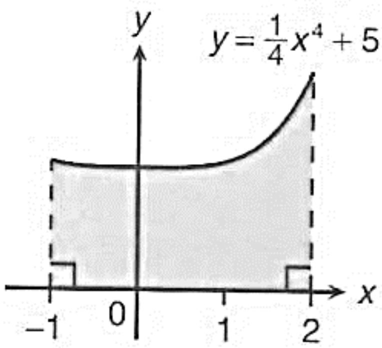
If $f(x) \leq 0$ for all x in the interval $[a, b]$, then the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ can be found by integrating $f(x)$ with respect to x from a to b,

i.e. Area $A = -\int_a^b f(x)\mathrm{d}x$



Example

1. Find the area of the region bounded by $y = \frac{1}{4}x^4 + 5$, the x-axis and the lines $x = -1$ and $x = 2$ as shown in the figure.



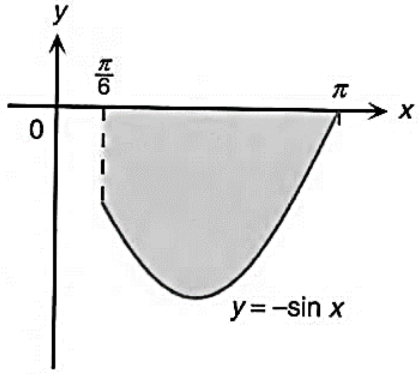
area $= \int_{-1}^{2} (\frac{1}{4}x^4 + 5)\,dx$

$= \left[\frac{1}{20}x^5 + 5x\right]_{-1}^{2}$

$= \frac{8}{5} + 10 - (-\frac{1}{20} - 5)$

$= \frac{333}{20}$

2. Find the area of the region bounded by $y=-\sin x$ and the x-axis between $x=\frac{\pi}{6}$ and $x=\pi$ as shown in the figure.



$\int_{\frac{\pi}{6}}^{\pi}(-\sin x)\,dx$

$=[\cos x]_{\frac{\pi}{6}}^{\pi}$

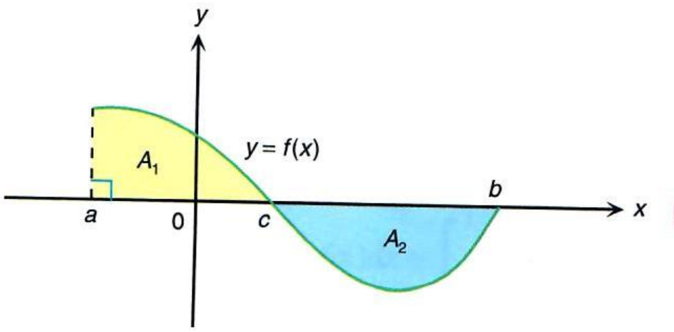
$=\cos\pi-\cos\frac{\pi}{6}$

$=-1-\frac{\sqrt{3}}{2}$

$\therefore$ required area

$=1+\frac{\sqrt{3}}{2}$

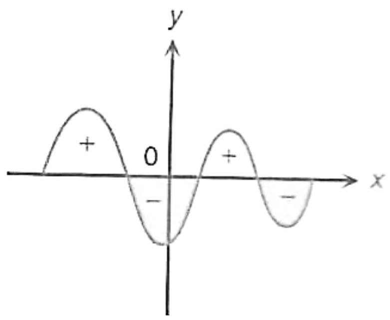
In general, a curve $y=\mathrm{f}(x)$ may not lie entirely above or below the x-axis as shown in the following figure.



In this case, the total area bounded by the graph of $y=\mathrm{f}(x)$, the x-axis and the lines $x=a$ and $x=b$ is given by:

$$A_1+A_2=\int_a^c \mathrm{f}(x)\mathrm{d}x+\left[-\int_c^b \mathrm{f}(x)\mathrm{d}x\right]$$

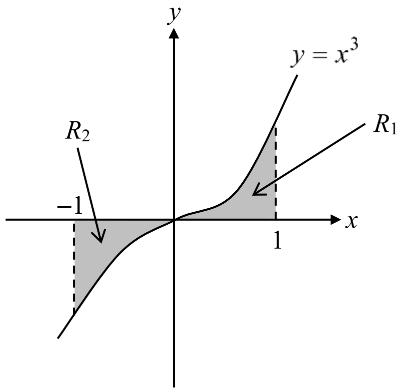
The area bounded by the graph of $y=\mathrm{f}(x)$ and the x-axis is given by definite integrals, as illustrated in the figure below:



Example

3. The figure shows the graph of $y = x^3$. Find the sum of the areas of the shaded region R_1 and R_2

$\int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$ X



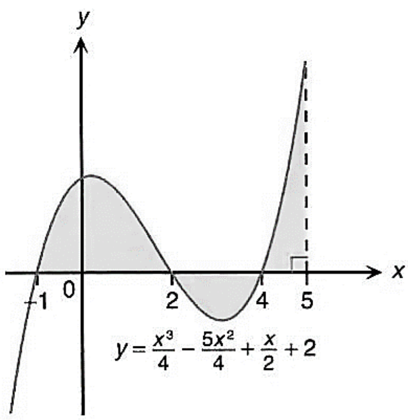
area $= -\int_{-1}^{0} x^3\,dx + \int_{0}^{1} x^3\,dx$

$= -\left[\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^4}{4}\right]_{0}^{1}$

$= \frac{1}{4} + \frac{1}{4}$

$= \frac{1}{2}$

4. Find the area of the shaded region in the following figure.



area

$= \int_{-1}^{2} \frac{x^3}{4} - \frac{5x^2}{4} + \frac{x}{2} + 2\,dx -$

$\int_{2}^{4} \frac{x^3}{4} - \frac{5x^2}{4} + \frac{x}{2} + 2\,dx +$

$\int_{4}^{5} \frac{x^3}{4} - \frac{5x^2}{4} + \frac{x}{2} + 2\,dx$

$= \left[\frac{x^4}{20} - \frac{5x^3}{12} + \frac{x^3}{6} + 2x\right]_{-1}^{2}$

$- \left[\frac{x^4}{20} - \frac{5x^3}{12} + \frac{x^3}{6} + 2x\right]_{2}^{4}$

$+ \left[\frac{x^4}{20} - \frac{5x^3}{12} + \frac{x^3}{6} + 2x\right]_{4}^{5}$

$= \frac{63}{16} - \left(-\frac{4}{3}\right) + \frac{91}{48}$

$= \frac{43}{6}$

5. The figure shows the graph of $y=(x-1)(x-2)$. Find the total area of the shaded regions.

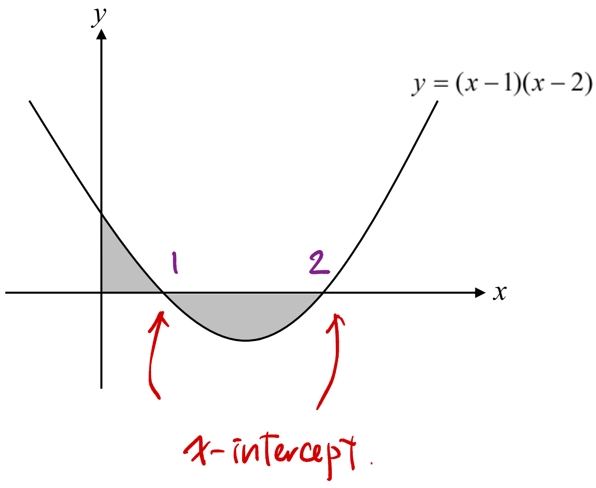
area

$$=\int_0^1 (x-1)(x-2)\,dx - \int_1^2 (x-1)(x-2)\,dx$$

$$=\int_0^1 (x^2-3x+2)\,dx - \int_1^2 (x^2-3x+2)\,dx$$

$$=\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_0^1 - \left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_1^2$$

$$=\frac{5}{6}-\left(-\frac{1}{6}\right)=1$$



經過 x-int, $+\to-$ or $-\to+$

6. Consider the curve $y=(x-1)(x-3)(x-5)$.
   (a) Find the x-intercepts of the curve.
   (b) Find the area bounded by the graph $y=(x-1)(x-3)(x-5)$, the x-axis and the lines $x=1$ and $x=5$.

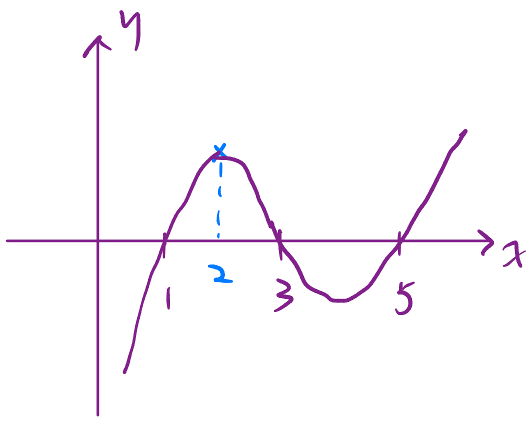
(a) 1, 3, 5

(b) put $x=2$,

$y=(2-1)(2-3)(2-5)$

$=1\cdot(-1)\cdot(-3)$

$=3,\ >0$



area

$$=\int_1^3 (x-1)(x-3)(x-5)\,dx - \int_3^5 (x-1)(x-3)(x-5)\,dx$$

$$=\int_1^3 (x^3-9x^2+23x-15) - \int_3^5 (x^3-9x^2+23x-15)\,dx$$

$$=\left[\frac{x^4}{4}-3x^3+\frac{23x^2}{2}-15x\right]_1^3 - \left[\frac{x^4}{4}-3x^3+\frac{23x^2}{2}-15x\right]_3^5$$

$$=4-(-4)$$

$$=8$$

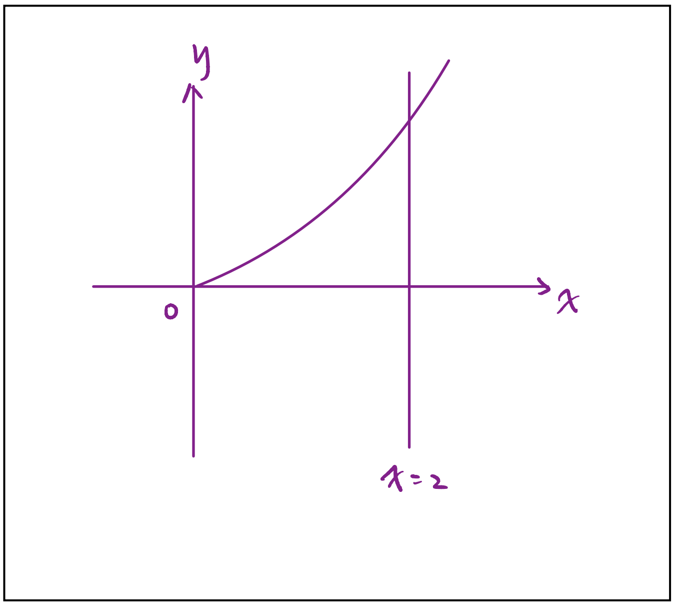
$\frac{d}{dx} 4^x = 4^x \cdot \ln 4$ $\int 4^x dx = \frac{4^x}{\ln 4} + c$

7. (a) Find the x-intercept(s) of the curve $y = 4^x - 1$.
   (b) Find the area of the region bounded by the curve $y = 4^x - 1$, the line $x = 2$ and the x-axis.

(a) put $y=0$, $4^x - 1 = 0$

$4^x = 1$

$x = 0$



(b) area

$= \int_0^2 (4^x - 1)\, dx$

$= \left[\frac{4^x}{\ln 4} - x\right]_0^2$

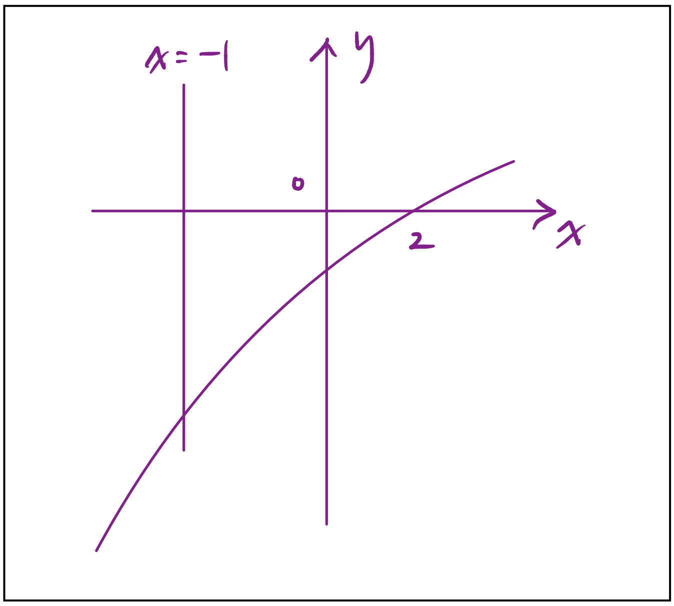
$= \frac{16}{\ln 4} - 2 - \frac{1}{\ln 4} + 0$

$= \frac{15}{\ln 4} - 2$

8. Find the area of the region bounded by the curve $y = 2^x - 4$, the line $x = -1$ and the x-axis.

put $y=0$, $2^x - 4 = 0$

$x = 2$



$\int_{-1}^2 (2^x - 4)\, dx$

$= \left[\frac{2^x}{\ln 2} - 4x\right]_{-1}^2$

$= \frac{2}{\ln 2} - 4 \cdot 2 - \left(\frac{2^{-1}}{\ln 2} + 4\right)$

$= \frac{3}{2\ln 2} - 4$ $\therefore$ required area $= 4 - \frac{3}{2\ln 2}$

B. Area between a Curve and the *y*-axis

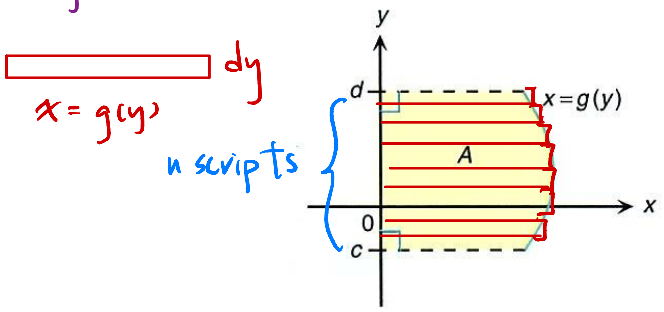
Let $x = \mathrm{g}(y)$ be a continuous function defined on the interval [c, d].

If $\mathrm{g}(y) \geq 0$ for all y in the interval [c, d], then the area bounded by the curve $x = \mathrm{g}(y)$, the y-axis and the lines y = c and y = d is given by:

Area $A = \int_c^d \mathrm{g}(y)\mathrm{d}y$

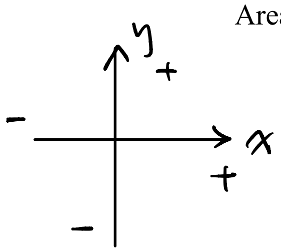
$\int_c^d x \, dy$

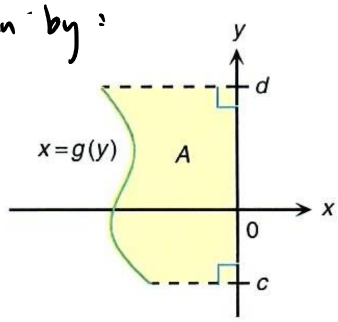
Sum , $n \to \infty$



If $\mathrm{g}(y) \leq 0$ for all y in the interval [c, d], then the area bounded by the curve $x = \mathrm{g}(y)$, the y-axis and the lines y = c and y = d is given by:

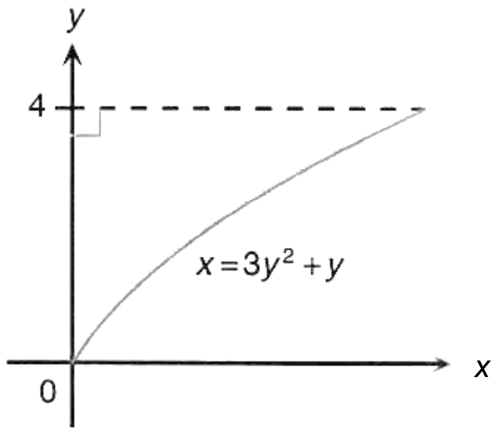
Area $A = -\int_c^d \mathrm{g}(y)\mathrm{d}y$





Example

8. Find the area of the region bounded by $x = 3y^2 + y$, the y-axis and the lines $y = 0$ and $y = 4$ as shown in the figure.



area

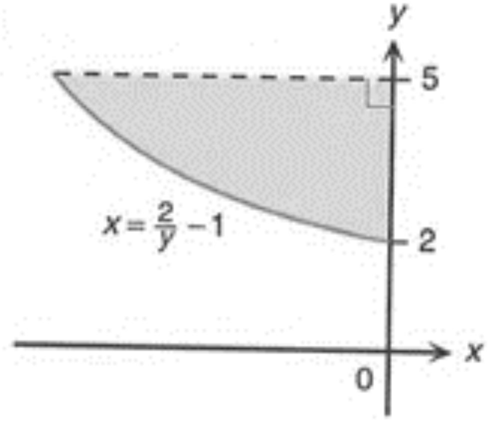
$= \int_0^4 (3y^2 + y)\, dy$

$= \left[y^3 + \frac{y^2}{2} \right]_0^4$

$= 64 + 8$

$= 72$

9. Find the area of the region bounded by $x = \frac{2}{y} - 1$, the y-axis and the lines $y = 2$ and $y = 5$ as shown in the figure.



area

$= -\int_2^5 \left(\frac{2}{y} - 1\right) dy$

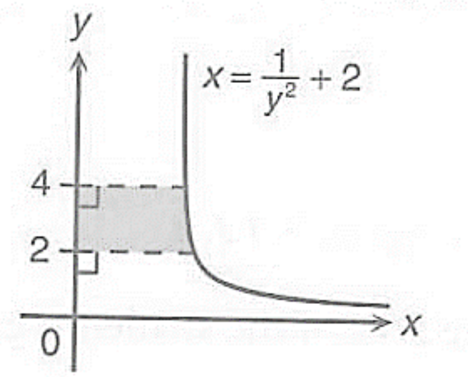
$= -\left[2\ln y - y\right]_2^5$

$= -(2\ln 5 - 5 - 2\ln 2 + 2)$

$= -\left(2\ln\frac{5}{2} - 3\right)$

$= 3 - 2\ln\frac{5}{2}$

10. Find the area of the region bounded by $x = \frac{1}{y^2} + 2$, the y-axis and the lines $y = 2$ and $y = 4$ as shown in the figure.



area

$= \int_2^4 \left(\frac{1}{y^2} + 2\right) dy$

$= \left[-y^{-1} + 2y\right]_2^4$

$= -\frac{1}{4} + 8 + \frac{1}{2} - 4$

$= \frac{17}{4}$

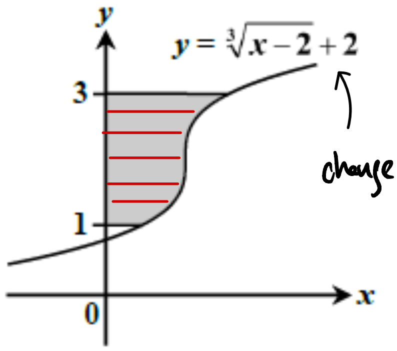
Check Your Concept

Example

11. Find the area of the shaded region in each of the following figures.

(a)

$\int_1^3 x\,dy$



change subject

$y = \sqrt[3]{x-2}+2$

$y-2 = \sqrt[3]{x-2}$

$(y-2)^3 = x-2$

$(y-2)^3+2 = x$

$dy \to d(y-2)$

area

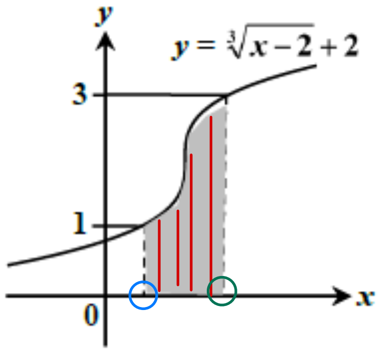
$= \int_1^3 [(y-2)^3+2]\,dy$

$= \left[\frac{(y-2)^4}{4} + 2y\right]_1^3$

$= \frac{1}{4}+6-\left(\frac{1}{4}+2\right)$

$= 4$

(b)



$\int_{○}^{○} y\,dx$

when $y=1$, $1 = \sqrt[3]{x-2}+2$

$-1 = \sqrt[3]{x-2}$

$-1 = x-2$

$x = 1$

when $y=3$, $3 = \sqrt[3]{x-2}+2$

$1 = \sqrt[3]{x-2}$

$1 = x-2$

$3 = x$

area

$= \int_1^3 \left(\sqrt[3]{x-2}+2\right) dx$

$= \left[\frac{3(x-2)^{\frac{4}{3}}}{4} + 2x\right]_1^3$

$= \frac{3}{4}+6-\left(\frac{3}{4}+2\right)$

$= 4$

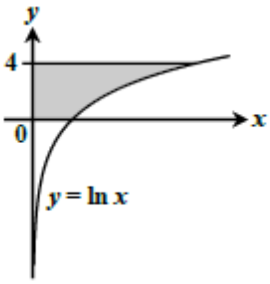
12. (a) Using integration by part, find $\int \ln x \, dx$.

$$\int \ln x \, dx = x\ln x - \int x \cdot \frac{1}{x} dx$$
$$= x\ln x - \int 1 \, dx$$
$$= x\ln x - x + C$$

(b) Find the area of the shaded region in each of the following figures.

(i)

$\int x \, dy$



$y = \ln x$

$e^y = x$

area

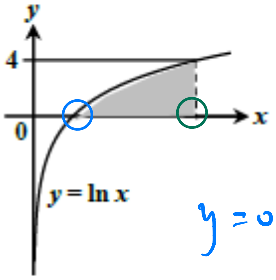
$= \int_0^4 e^y \, dy$

$= [e^y]_0^4$

$= e^4 - 1$

(ii)

$\int y \, dx$



$y = 0$, $\ln x = 0$

$x = 1$

$y = 4$, $\ln x = 4$

$x = e^4$

area

$= \int_1^{e^4} \ln x \, dx$

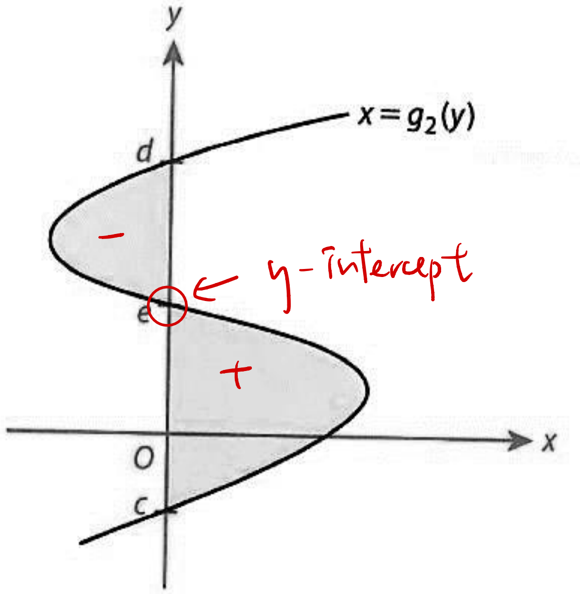
$= [x\ln x - x]_1^{e^4}$

$= e^4 \ln e^4 - e^4 - (0 - 1)$

$= 4e^4 - e^4 + 1$

$= 3e^4 + 1$

In general, a curve $x = g_2(y)$ lies on both sides of the y-axis.

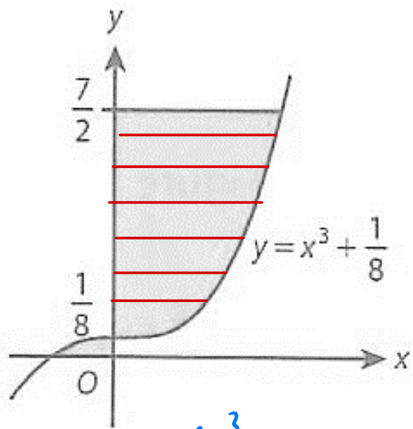


Thus, the total area bounded by the graph of $x = g_2(y)$, the y-axis and the lines $y = c$ and $y = d$ is given by:

$$\textbf{Area of the shaded region} = \int_c^e g_2(y)\,\mathrm{d}\overset{y}{\cancel{x}} + \left[-\int_d^e g_2(y)\,\mathrm{d}\overset{y}{\cancel{x}}\right]$$

Example

13. Find the area of the shaded region bounded by the curve $y = x^3 + \frac{1}{8}$, the y-axis, the x-axis and the line $y = \frac{7}{2}$.



$$\text{area} = -\int_0^{\frac{1}{8}} \left(y - \tfrac{1}{8}\right)^{\frac{1}{3}} dy + \int_{\frac{1}{8}}^{\frac{7}{2}} \left(y - \tfrac{1}{8}\right)^{\frac{1}{3}} dy$$

$$= -\left[\frac{3\left(y - \frac{1}{8}\right)^{\frac{4}{3}}}{4}\right]_0^{\frac{1}{8}} + \left[\frac{3\left(y - \frac{1}{8}\right)^{\frac{4}{3}}}{4}\right]_{\frac{1}{8}}^{\frac{7}{2}}$$

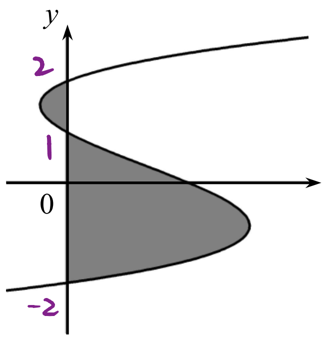
$$= -\left(0 - \frac{3}{64}\right) + \left(\frac{243}{64} - 0\right)$$

$$= \frac{123}{32}$$

$$y - \frac{1}{8} = x^3$$

$$\left(y - \frac{1}{8}\right)^{\frac{1}{3}} = x$$

14. In the figure, find the area of the shaded region bounded by the curve $x = y^3 - y^2 - 4y + 4$ and the y-axis.



put $x=0$, $y^3 - y^2 - 4y + 4 = 0$

$y^2(y-1) - 4(y-1) = 0$

$(y-1)(y^2-4) = 0$

$(y-1)(y+2)(y-2) = 0$

$y = 1, -2, 2$

area

$= \int_{-2}^{1} (y^3 - y^2 - 4y + 4)\,dy - \int_{1}^{2} (y^3 - y^2 - 4y + 4)\,dy$

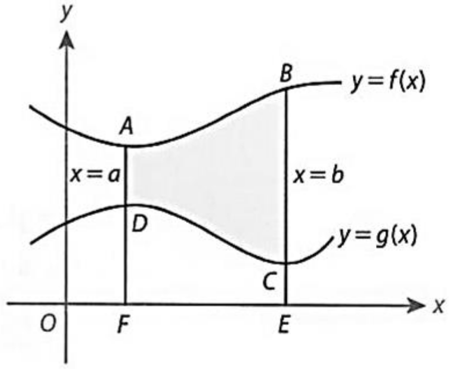
$= \left[\frac{y^4}{4} - \frac{y^3}{3} - 2y^2 + 4y\right]_{-2}^{1} - \left[\frac{y^4}{4} - \frac{y^3}{3} - 2y^2 + 4y\right]_{1}^{2}$

$= \frac{45}{4} - \left(-\frac{7}{12}\right)$

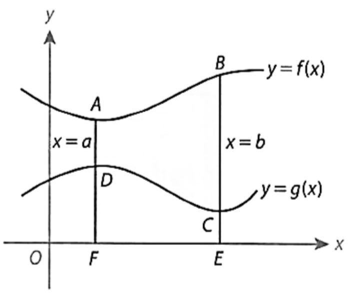
$= \frac{71}{6}$

C. Area between Two Curves

Suppose $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ are two continuous functions on the interval $[a,b]$ with $\mathrm{f}(x) \ge \mathrm{g}(x) \ge 0$.



Consider the area of the region bounded by the curves and the vertical lines $x = a$ and $x = b$ as follows:



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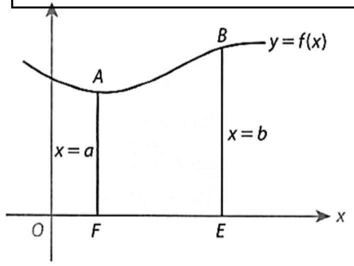
$\int_a^b f(x)dx$



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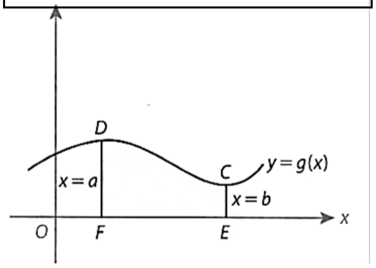
$\int_a^b g(x)dx$



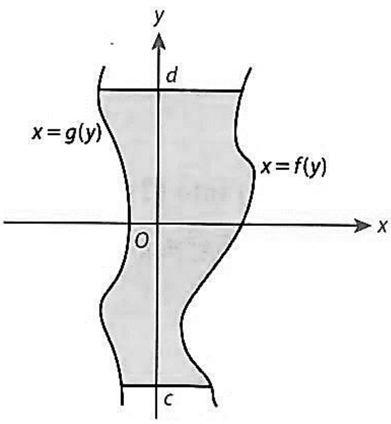
$$\text{Area} = \int_a^b [\mathrm{f}(x) - \mathrm{g}(x)]\,\mathrm{d}x$$

上減下

Similarly, when $\mathrm{f}(y)$ and $\mathrm{g}(y)$ are continuous functions on the interval $[c,d]$ with $\mathrm{f}(y) \ge \mathrm{g}(y)$, the area of the region bounded by the curves $x = \mathrm{f}(y)$ and $x = \mathrm{g}(y)$ and the horizontal lines $y = c$ and $y = d$ can be found using the following formula:

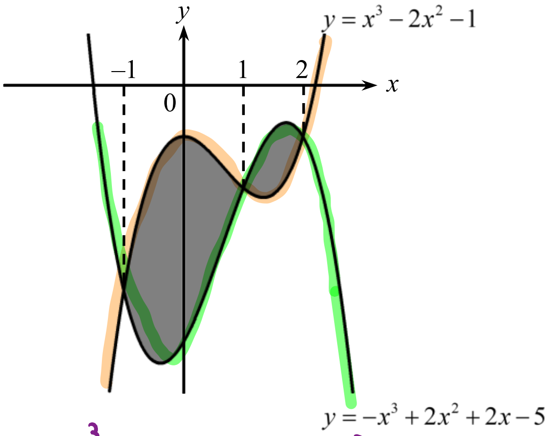
$$\text{Area} = \int_c^d [\mathrm{f}(y) - \mathrm{g}(y)]\,\mathrm{d}y$$

右減左



Example

15. In the figure, find the area of the shaded region bounded by the curves $y=-x^3+2x^2+2x-5$ and $y=x^3-2x^2-1$.



Area

$= \int_{-1}^{1} [x^3-2x^2-1-(-x^3+2x^2+2x-5)]\,dx + \int_{1}^{2} [-x^3+2x^2+2x-5-(x^3-2x^2-1)]\,dx$

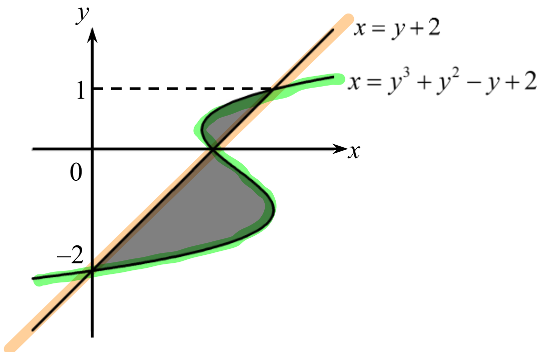
$= \int_{-1}^{1} (2x^3-4x^2-2x+4)\,dx + \int_{1}^{2} (-2x^3+4x^2+2x-4)\,dx$

$= \left[\frac{x^4}{2}-\frac{4x^3}{3}-x^2+4x\right]_{-1}^{1} + \left[-\frac{x^4}{2}+\frac{4x^3}{3}+x^2-4x\right]_{1}^{2}$

$= \frac{16}{3}-\left(-\frac{5}{6}\right)$

$= \frac{37}{6}$

16. In the figure, find the area of the shaded region bounded by the curve $x=y^3+y^2-y+2$ and the line $x=y+2$.



Area

$= \int_{-2}^{0} (y^3+y^2-y+2-y-2)\,dy + \int_{0}^{1} (y+2-y^3-y^2+y-2)\,dy$

$= \int_{-2}^{0} (y^3+y^2-2y)\,dy + \int_{0}^{1} (-y^3-y^2+2y)\,dy$

$= \left[\frac{y^4}{4}-\frac{y^3}{3}-y^2\right]_{-2}^{0} + \left[-\frac{y^4}{4}-\frac{y^3}{3}+y^2\right]_{0}^{1}$

$= \frac{8}{3}-\left(-\frac{5}{12}\right)$

$= \frac{37}{12}$

17. Find the area of the region bounded by the curve $y=-x^2-2x+3$ and the line $y=x+3$.

Consider $-x^2-2x+3 = x+3$

$0 = x^2+3x$

$0 = x(x+3)$

$x = 0$ or -3

$x=-1$

$y = -(-1)^2-2(-1)+3$

$= -1+2+3$

$= 4$

$y = -1+3$

$= 2$

area $= \int_{-3}^{0} [-x^2-2x+3-(x+3)]\,dx$

$= \int_{-3}^{0} (-x^2-3x)\,dx$

$= \left[-\frac{x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0}$

$= -\left[-\frac{(-3)^3}{3} - \frac{3(-3)^2}{2}\right]$

$= \frac{9}{2}$

$y=x-4$

18. Find the area of the region bounded by the curve $y=x^3-3x-4$ and the line $x-y-4=0$

Consider $x^3-3x-4 = x-4$

$x^3-4x = 0$

$x(x^2-4) = 0$

$x = 0, 2, -2$

$x=1$

$y = 1^3-3-4 = -6$

$y = 1-4 = -3$

$x=-1$

$y = (-1)^3-3(-1)-4$

$= -2$

$y = -1-4$

$= -5$

area $= \int_{-2}^{0} [x^3-3x-4-(x-4)]\,dx +$

$\int_{0}^{2} [x-4-(x^3-3x-4)]\,dx$

$= \int_{-2}^{0} (x^3-4x)\,dx + \int_{0}^{2} (4x-x^3)\,dx$

$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} + \left[2x^2 - \frac{x^4}{4}\right]_{0}^{2}$

$= -(4-8) + (8-4)$

$= 8$

19. (a) Using integration by parts, find $\int xe^{-x}\mathrm{d}x$.

(b) Consider the curve $C: y = xe^{-x}$ and the line $L: y = \dfrac{x}{e^2}$.

(i) Find the x-coordinates of the points of intersection of C and L.

(ii) Find the area of the region bounded by C and L.

(a) $\int xe^{-x}dx$

$= -\int x\,de^{-x}$

$= -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + C$

$x=1,\quad y = 1\cdot e^{-1} = \frac{1}{e}$

$y = \frac{1}{e^2}$

(b) (i) Consider $xe^{-x} = \dfrac{x}{e^2}$

$xe^{-x} - \dfrac{x}{e^2} = 0$

$x\left(e^{-x} - \dfrac{1}{e^2}\right) = 0$

$x = 0$ or 2

(ii) area $= \int_0^2 \left(xe^{-x} - \dfrac{x}{e^2}\right) dx$

$= \left[-xe^{-x} - e^{-x} - \dfrac{1}{e^2}\dfrac{x^2}{2}\right]_0^2$

$= -2e^{-2} - e^{-2} - 2e^{-2} - (0 - 1 - 0)$

$= 1 - 5e^{-2}$

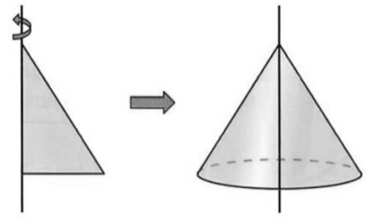
$= 1 - \dfrac{5}{e^2}$

9.2 Volumes of Solids of Revolution

A solid generated by revolving a plane region about a straight line (the **axis of revolution**) is called a **solid of revolution**.

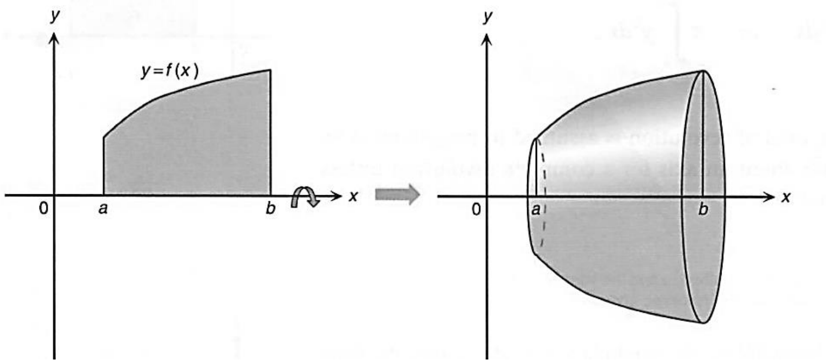
horizontal / vertical

Example:

	*Revolving a right-angled triangle about one of its adjacent sides to the right angle results in a cone.*

A. Volumes of Solids of Revolution Generated by Revolving about a Coordinate Axis

Consider the plane region bounded by a curve $y = f(x)$, the x-axis and two vertical lines $x = a$ and $x = b$ as shown in the figure. Then, we revolve the region about the x-axis through a complete revolution to generate a solid.



Suppose the interval $[a,b]$ is divided into n sub-intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ of equal width Δx. Choose a point c_k, where $k = 1, 2, \ldots, n$, from each sub-interval $[x_{k-1}, x_k]$.

Consider the slice with radius $f(c_k)$ and thickness Δx. We denote the volume of this slice as ΔV_k.

Then $\Delta V_k = \pi[f(c_k)]^2 \Delta x$.

By summing up the volumes of the n slices, we can find the volume of the solid of revolution.

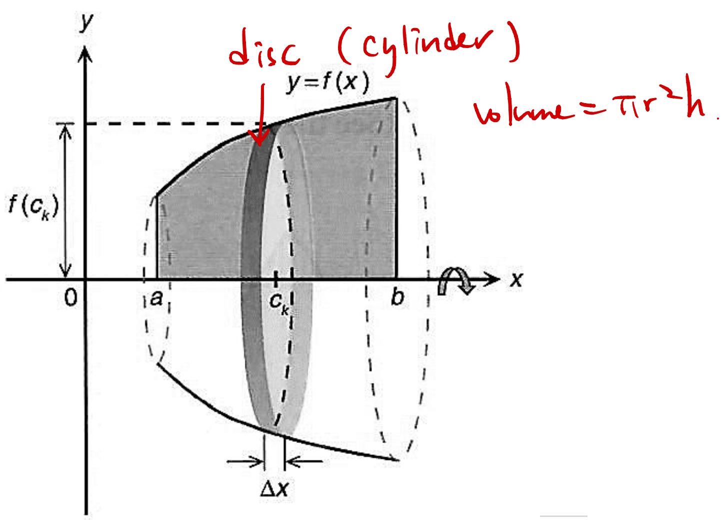
Hence,

Volume of the solid is given by:

$$V = \lim_{\Delta x \to 0} \sum_{k=1}^{n} \pi [f(c_k)]^2 \Delta x$$

$$= \int_a^b \pi \underbrace{[f(x)]^2}_{r^2} \underbrace{dx}_{h}$$

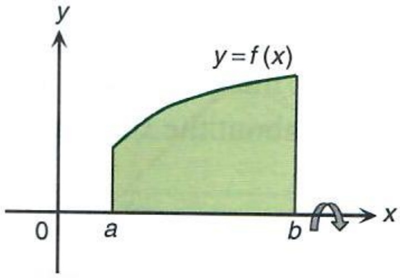
$$= \pi \int_a^b [f(x)]^2 \, dx$$



Theorem 1 Disc Method – Revolution about the *x*-axis

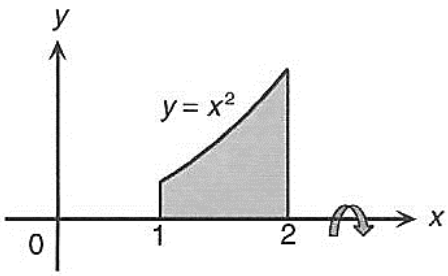
For any function $y = f(x)$ defined on the interval $[a,b]$, the volume of the solid generated by revolving the region bounded by $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$ about the x-axis is given by

$$V = \pi\int_a^b [f(x)]^2\,dx \qquad \text{or} \qquad V = \pi\int_a^b y^2\,dx$$



Example

20. In the figure, the region bounded by the parabola $y = x^2$, the x-axis, the lines $x = 1$ and $x = 2$ is revolved about the x-axis. Find the volume of the solid of revolution.



volume

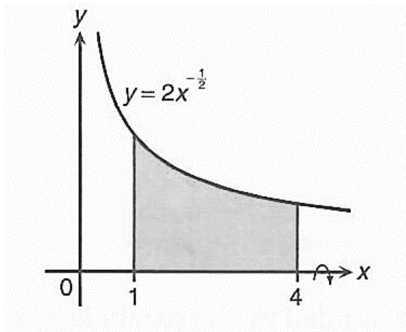
$= \int_1^2 \pi \cdot y^2\,dx$

$= \int_1^2 \pi\, x^4\,dx$

$= \pi \cdot \left[\frac{x^5}{5}\right]_1^2$

$= \frac{31\pi}{5}$

21. In the figure, the region bounded by the curve $y = 2x^{-\frac{1}{2}}$, the x-axis, the lines $x = 1$ and $x = 4$ is revolved about the x-axis. Find the volume of the solid of revolution.



volume

$= \int_1^4 \pi y^2\,dx$

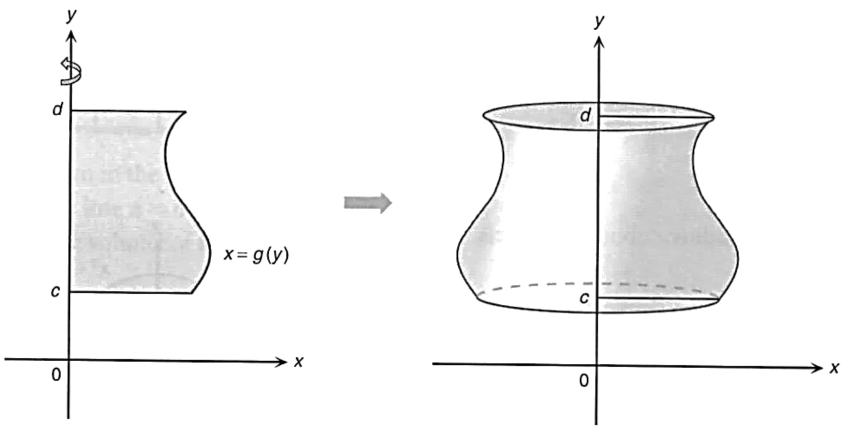
$= \int_1^4 \pi\,(2x^{-\frac{1}{2}})^2\,dx$

$= 4\pi\int_1^4 x^{-1}\,dx$

$= 4\pi\,[\ln x]_1^4$

$= 4\pi \ln 4$

Consider the plane region bounded by a curve $x = g(y)$, the y-axis and two horizontal lines $y = c$ and $y = d$ as shown in the figure. Then the region is revolved about the y-axis through a complete revolution to generate a solid of revolution.



Suppose the interval $[c,d]$ is divided into n sub-intervals $[y_0, y_1]$, $[y_1, y_2]$, …, $[y_{n-1}, y_n]$ of equal width Δy. Choose a point t_k, where $k = 1, 2, \ldots, n$, from each sub-interval $[y_{k-1}, y_k]$.
Consider the slice with radius $g(t_k)$ and thickness Δy. We denote the volume of this slice as ΔV_k.

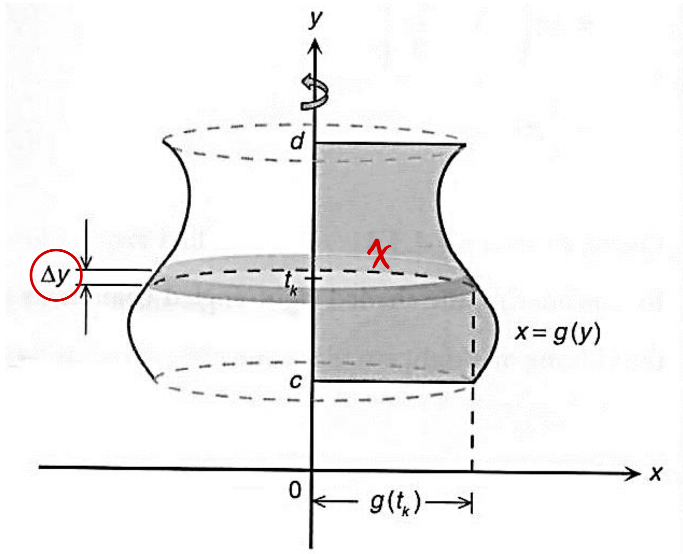
Then $\Delta V_k = \pi[g(t_k)]^2 \Delta y$.

Volume of the solid is given by:

$$V = \lim_{\Delta y \to 0} \sum_{k=1}^{n} \pi[g(t_k)]^2 \Delta y$$

$$= \int_c^d \pi[g(y)]^2 \, dy$$

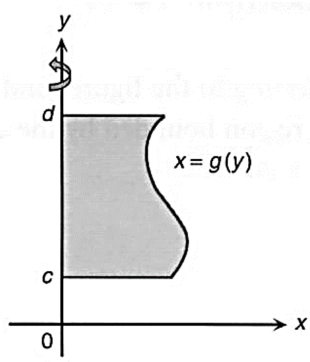
$$= \pi \int_c^d [g(y)]^2 \, dy$$



Theorem 2 Disc Method – Revolution about the *y*-axis

For any function $x = g(y)$ defined on the interval $[c,d]$, the volume of the solid generated by revolving the region bounded by $x = g(y)$, the y-axis, the lines $y = c$ and $y = d$ about the y-axis is given by

$$V = \pi \int_c^d [g(y)]^2 \, dy \qquad \text{or} \qquad V = \pi \int_c^d x^2 \, dy$$



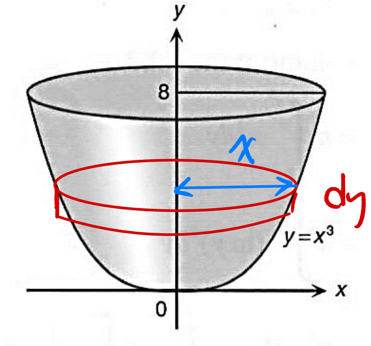
Example

22. Referring to the figure, find the volume of the solid generated by revolving the region bounded by the curve $y = x^3$, the y-axis and the line $y = 8$ about the y-axis.

$y = x^3$

$y^{\frac{2}{3}} = x^2$

$$\text{volume} = \int_0^8 \pi x^2 dy = \int_0^8 \pi y^{\frac{2}{3}} dy = \pi \left[\frac{3y^{\frac{5}{3}}}{5} \right]_0^8 = \frac{96\pi}{5}$$



23. Find the volume of the solid of revolution generated by revolving the region (as shown) bounded by the y-axis, the curve $y = 2x^2 - 1$ and the line $y = 1$ about the y-axis.

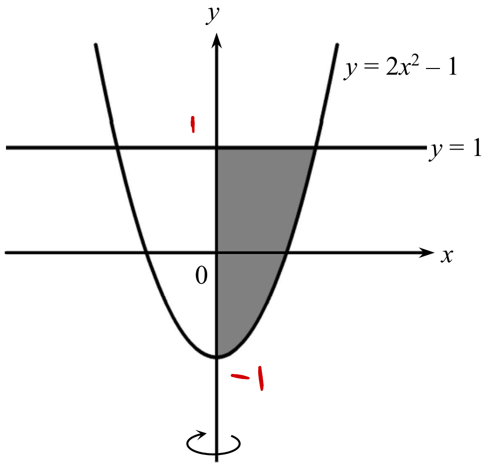
$y = 2x^2 - 1$

$\frac{y+1}{2} = x^2$

put $x = 0$,

$y = -1$

$$\text{volume} = \pi \int_{-1}^{1} \frac{y+1}{2} dy = \frac{\pi}{2} \left[\frac{y^2}{2} + y \right]_{-1}^{1} = \frac{\pi}{2} \left(\frac{1}{2} + 1 - \frac{1}{2} + 1 \right) = \pi$$



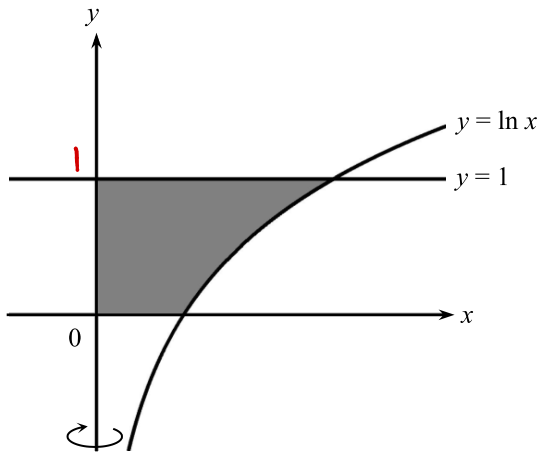
24. Find the volume of the solid of revolution generated by revolving the region (as shown) bounded by the x-axis, the y-axis, the curve $y = \ln x$ and the line $y = 1$ about the y-axis.

$y = \ln x$

$e^y = x$

$e^{2y} = x^2$

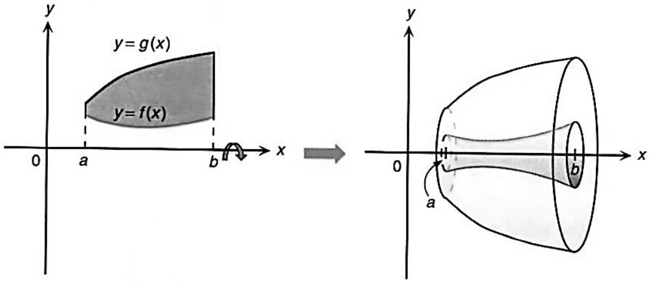
$$\text{volume} = \pi \int_0^1 x^2 dy = \pi \int_0^1 e^{2y} dy = \pi \left[\frac{e^{2y}}{2} \right]_0^1 = \frac{\pi}{2} (e^2 - 1)$$



空心

Hollow Solid of Revolution

If a plane region bounded by two curves $y = \mathrm{f}(x)$, $y = \mathrm{g}(x)$, the lines $x = a$ and $x = b$ is revolved about the x-axis, a **hollow solid** is generated.



外 - 內 / 遠 - 近

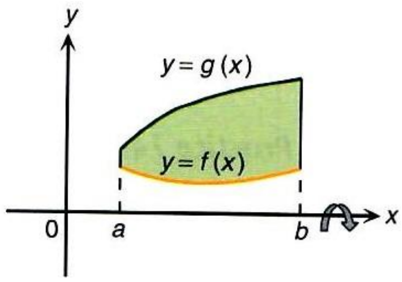
Obviously, the volume V of the hollow solid is given by: $\neq [g(x) - f(x)]^2$

$$V = \pi\int_a^b [\mathrm{g}(x)]^2\,\mathrm{d}x - \pi\int_a^b [\mathrm{f}(x)]^2\,\mathrm{d}x = \pi\int_a^b \left([\mathrm{g}(x)]^2 - [\mathrm{f}(x)]^2\right)\mathrm{d}x$$

Theorem 3 Disc Method – Hollow Solids about the x-axis

For the curves $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ that satisfy $0 \le \mathrm{f}(x) \le \mathrm{g}(x)$ for $a \le x \le b$, the volume of a solid generated by revolving a region bounded by two curves $y = \mathrm{f}(x)$, $y = \mathrm{g}(x)$, the lines $x = a$ and $x = b$ about the x-axis is given by

$$V = \pi\int_a^b \left([\mathrm{g}(x)]^2 - [\mathrm{f}(x)]^2\right)\mathrm{d}x$$



Example

25. Referring to the figure, find the volume of the solid generated by revolving the region bounded by the curve $y = 2x - x^2$ and the line $y = \dfrac{x}{2}$ about the x-axis.

$2x - x^2 = \frac{x}{2}$

$4x - 2x^2 = x$

$3x - 2x^2 = 0$

$x(3 - 2x) = 0$

$x = 0$ or $\frac{3}{2}$

volume $\pi y^2 dx$

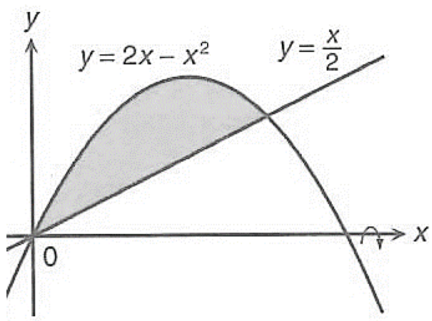
$= \pi \int_0^{\frac{3}{2}} [(2x - x^2)^2 - (\frac{x}{2})^2]\, dx$

$= \pi \int_0^{\frac{3}{2}} (4x^2 - 4x^3 + x^4 - \frac{x^2}{4})\, dx$

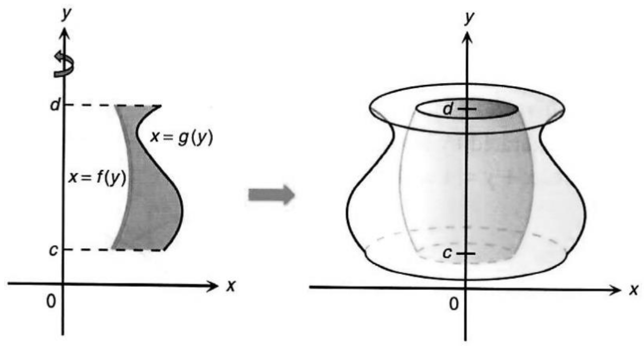
$= \pi \int_0^{\frac{3}{2}} (x^4 - 4x^3 + \frac{15}{4}x^2)\, dx$

$= \pi \left[\frac{x^5}{5} - x^4 + \frac{5}{4}x^3 \right]_0^{\frac{3}{2}}$

$= \frac{27}{40}\pi$



Similarly, if a plane region bounded by two curves $x = \mathrm{f}(y)$ and $x = \mathrm{g}(y)$, the lines $y = c$ and $y = d$ is revolved about the y-axis, a hollow solid is generated.

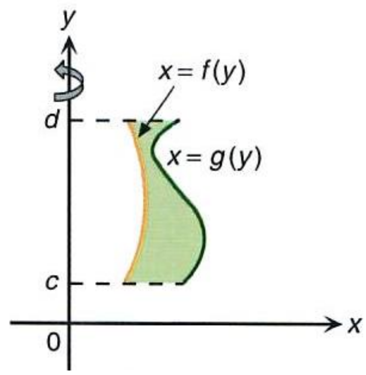


Theorem 4 Disc Method – Hollow Solids about the *y*-axis

For the curves $x = \mathrm{f}(y)$ and $x = \mathrm{g}(y)$ that satisfy $0 \le \mathrm{f}(y) \le \mathrm{g}(y)$ or $\mathrm{g}(y) \le \mathrm{f}(y) \le 0$ for $c \le y \le d$, the volume of a solid generated by revolving a region bounded by two curves $x = \mathrm{f}(y)$, $x = \mathrm{g}(y)$, the lines $y = c$ and $y = d$ about the y-axis is given by

$$V = \pi \int_c^d \left([\mathrm{g}(y)]^2 - [\mathrm{f}(y)]^2 \right) dy$$

$\neq [g(y) - f(y)]^2$



Example

26. Find the volume of the solid generated by revolving the shaded region bounded by the curves $x + y^2 = 1$ and $x = \sqrt{1 - y^2}$ about the y-axis.

volume

$= \pi \int_{-1}^{1} [(1-y^2) - (1-y^2)^2] \, dy$

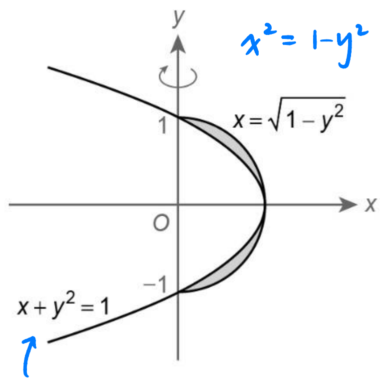
$= \pi \int_{-1}^{1} (1 - y^2 - 1 + 2y^2 - y^4) \, dy$

$= \pi \int_{-1}^{1} (y^2 - y^4) \, dy$

$= \pi \left[\frac{y^3}{3} - \frac{y^5}{5} \right]_{-1}^{1}$

$= \pi \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5} \right)$

$= \frac{4\pi}{15}$



$x^2 = 1 - y^2$

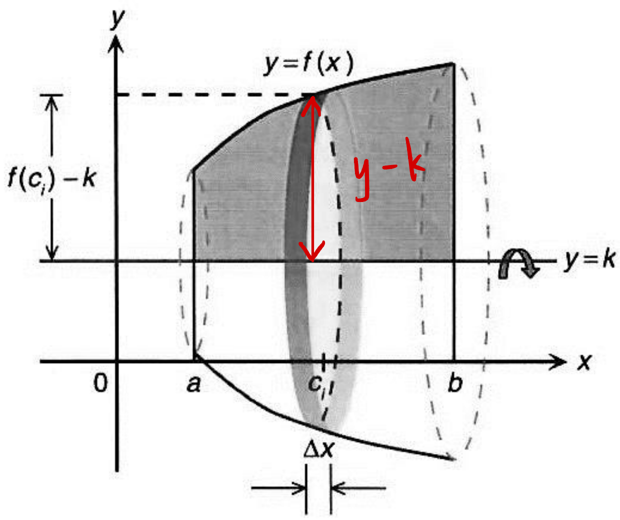
$x = 1 - y^2$

$x^2 = (1-y^2)^2$

vertical / horizontal

B. Volume of Solids of Revolution by Revolving about a Line Parallel to a Coordinate Axis

As shown in the figure, the plane region bounded by the curve $y = \mathrm{f}(x)$, the lines $x = a$ and $x = b$ is revolved about the line $y = k$ (which is parallel to the x-axis) to generate a hollow solid.

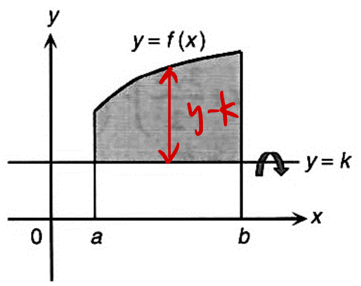


Theorem 5 Disc Method – Revolution about an Axis Parallel to the *x*-axis

For any function $y = \mathrm{f}(x)$ defined on the interval $[a,b]$, the volume of the solid generated by revolving the region bounded by $y = \mathrm{f}(x)$, the lines $y = k$, $x = a$ and $x = b$ about the line $y = k$ is given by

$$V = \pi\int_a^b [\mathrm{f}(x) - k]^2 \mathrm{d}x \qquad \text{or} \qquad V = \pi\int_a^b (y-k)^2 \mathrm{d}x$$

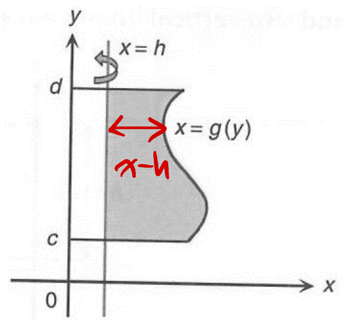
$$\neq \pi \int_a^b [f(x)^2 - k^2]\, dx$$



Theorem 6 Disc Method – Revolution about an Axis Parallel to the *y*-axis

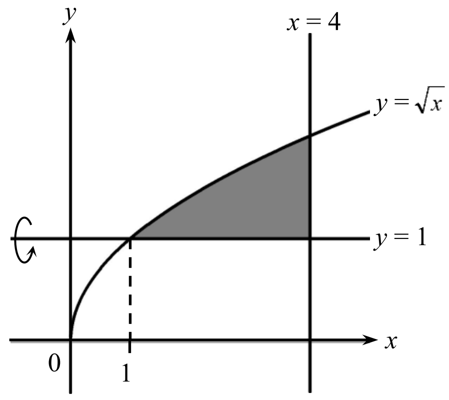
For any function $x = \mathrm{g}(y)$ defined on the interval $[c,d]$, the volume of the solid generated by revolving the region bounded by $x = \mathrm{g}(y)$, the lines $x = h$, $y = c$ and $y = d$ about the line $x = h$ is given by

$$V = \pi\int_c^d [\mathrm{g}(y) - h]^2 \mathrm{d}y \qquad \text{or} \qquad V = \pi\int_c^d (x-h)^2 \mathrm{d}y$$



Example

27. Find the volume of the solid of revolution generated by revolving the region (as shown) bounded by the curve $y=\sqrt{x}$ and the lines $y = 1$ and $x = 4$ about the line $y = 1$.



volume $= \int_1^4 \pi (y-1)^2\,dx$

$= \int_1^4 \pi (\sqrt{x}-1)^2\,dx$

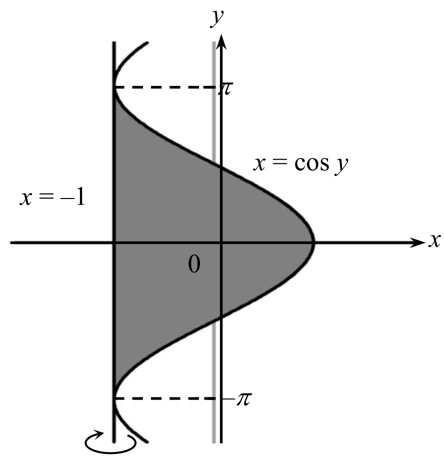
$= \pi \int_1^4 (x - 2\sqrt{x} + 1)\,dx$

$= \pi \left[\frac{x^2}{2} - \frac{4x^{\frac{3}{2}}}{3} + x \right]_1^4$

$= \pi \left(8 - \frac{32}{3} + 4 - \frac{1}{2} + \frac{4}{3} - 1 \right)$

$= \frac{7\pi}{6}$

28. Find the volume of the solid of revolution generated by revolving the region (as shown) bounded by the line $x = -1$ and the curve $x = \cos y$ for $-\pi \le x \le \pi$ about the line $x = -1$.



volume $= \pi \int_{-\pi}^{\pi} [x-(-1)]^2\,dy$

$= \pi \int_{-\pi}^{\pi} (\cos y + 1)^2\,dy$

$= \pi \int_{-\pi}^{\pi} (\cos^2 y + 2\cos y + 1)\,dy$

$= \pi \int_{-\pi}^{\pi} \left(\frac{1+\cos 2y}{2} + 2\cos y + 1 \right) dy$

$= \pi \int_{-\pi}^{\pi} \left(\frac{3}{2} + 2\cos y + \frac{1}{2}\cos 2y \right) dy$

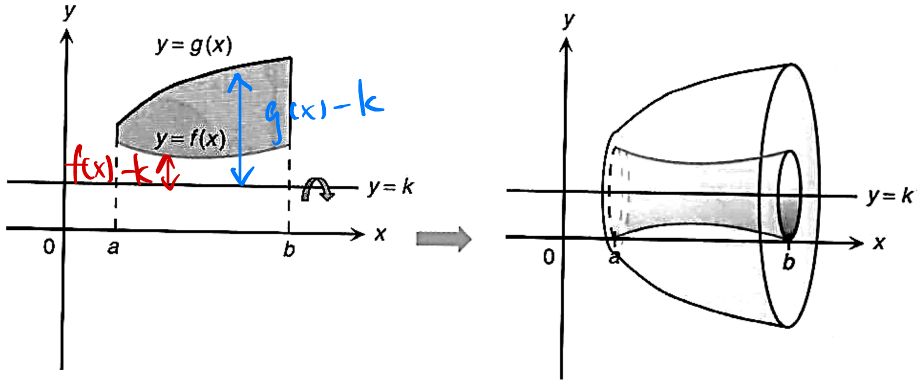
$= \pi \left[\frac{3}{2}y + 2\sin y + \frac{1}{4}\sin 2y \right]_{-\pi}^{\pi}$

$= \pi \left[\frac{3}{2}\pi - \left(-\frac{3}{2}\pi\right) \right]$

$= 3\pi^2$

Hollow Solid of Revolution

As shown in the figure, the plane region bounded by the curves $y = \mathrm{g}(x)$ and $y = \mathrm{f}(x)$, the lines $x = a$ and $x = b$ is revolved about the line $y = k$ (which is parallel to the x-axis) to generate a hollow solid.



Volume of the hollow solid

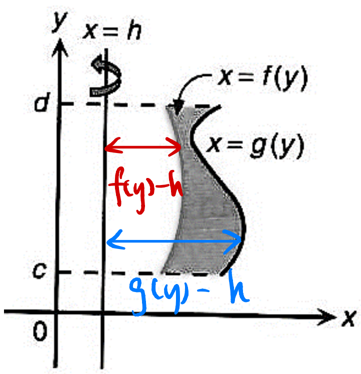
$$= \pi\int_a^b [\mathrm{g}(x)-k]^2\,\mathrm{d}x - \pi\int_a^b [\mathrm{f}(x)-k]^2\,\mathrm{d}x$$

$$= \pi\int_a^b \left\{[\mathrm{g}(x)-k]^2 - [\mathrm{f}(x)-k]^2\right\}\mathrm{d}x$$

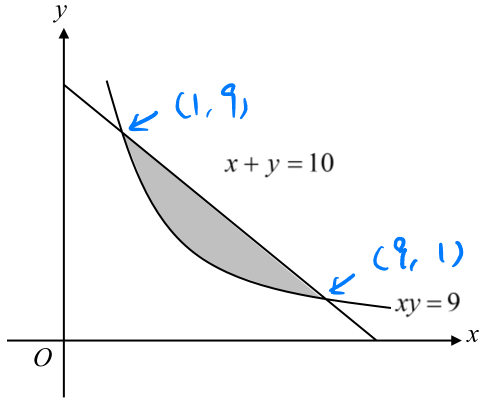
Similarly, if the plane region bounded by the curves $x = \mathrm{g}(y)$ and $x = \mathrm{f}(y)$, the lines $y = c$ and $y = d$ is revolved about the line $x = h$ (which is parallel to the y-axis) to generate a hollow solid, we have:

Volume of the hollow solid

$$= \pi\int_c^d \left\{[\mathrm{g}(y)-h]^2 - [\mathrm{f}(y)-h]^2\right\}\mathrm{d}y$$



29.



$x(10 - x) = 9$

$10x - x^2 = 9$

$x^2 - 10x + 9 = 0$

$x = 1$ or 9

The figure shows the graphs of $x + y = 10$ and $xy = 9$.

(a) Find the volume of the solid formed if the shaded region is revolved about the x-axis.

(b) Find the volume of the solid formed when the shaded region is revolved about the line $y = -1$.

(a) volume $= \pi \int_1^9 \left[(10 - x)^2 - \left(\frac{9}{x}\right)^2\right] dx$

$= \pi \int_1^9 \left(100 - 20x + x^2 - \frac{81}{x^2}\right) dx$

$= \pi \left[100x - 10x^2 + \frac{x^3}{3} + \frac{81}{x}\right]_1^9$

$= \pi \left[900 - 810 + 243 + 9 - \left(100 - 10 + \frac{1}{3} + 81\right)\right]$

$= \frac{512\pi}{3}$

(b) volume $= \pi \int_1^9 \left[(10 - x + 1)^2 - \left(\frac{9}{x} + 1\right)^2\right] dx$

$= \pi \int_1^9 \left[121 - 22x + x^2 - \left(\frac{81}{x^2} + \frac{18}{x} + 1\right)\right] dx$

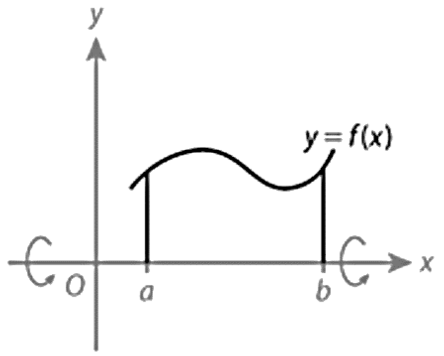
$= \pi \int_1^9 \left(120 - 22x + x^2 - \frac{81}{x^2} - \frac{18}{x}\right) dx$

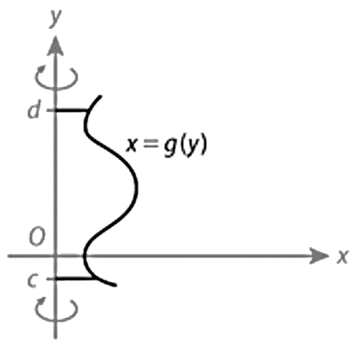
$= \pi \left[120x - 11x^2 + \frac{x^3}{3} + \frac{81}{x} - 18\ln x\right]_1^9$

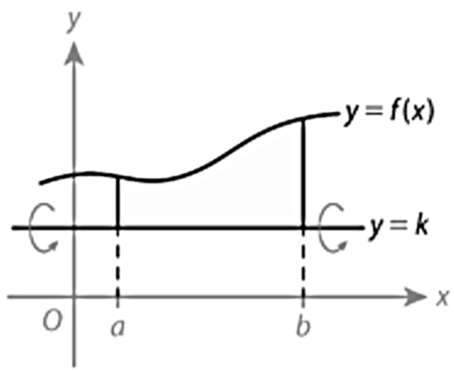
$= \pi \left(\frac{752}{3} - 18\ln 9\right)$

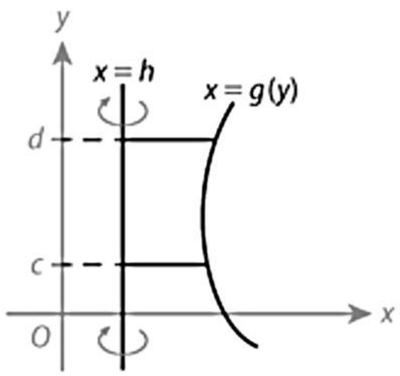
Summary: Volume of Solids of Revolution

A. Solid – 1 square [ ]^2

<table>
<tr><td colspan="2">Volume of a solid of revolution about a coordinate axis</td></tr>
<tr><td>About the x-axis


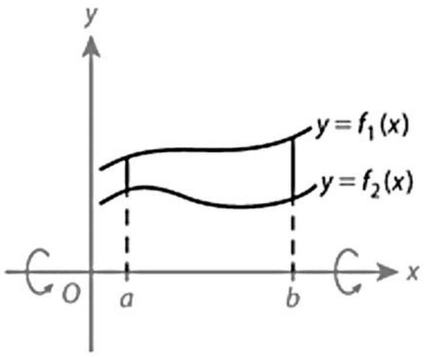
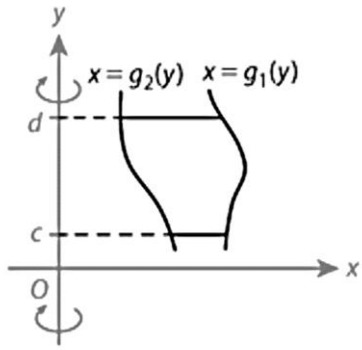
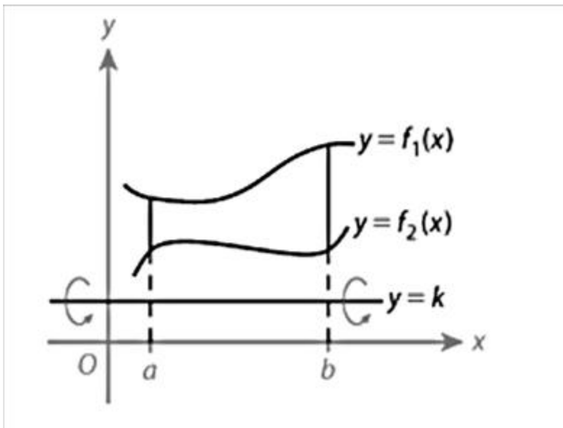
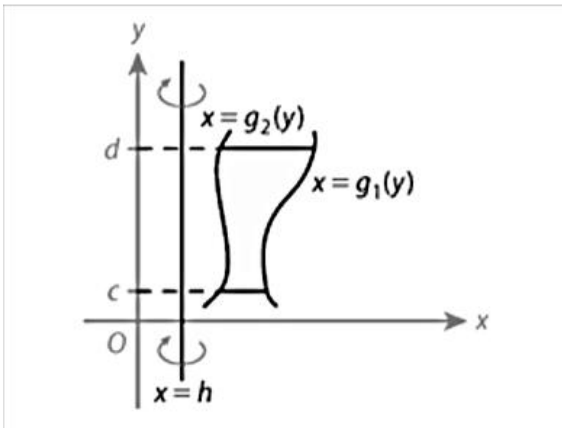
$\text{Volume} = \pi\int_a^b [\mathrm{f}(x)]^2\,\mathrm{d}x$</td><td>About the y-axis


$\text{Volume} = \pi\int_c^d [\mathrm{g}(y)]^2\,\mathrm{d}y$</td></tr>
<tr><td colspan="2">Volume of a solid of revolution about a line parallel to a coordinate axis</td></tr>
<tr><td>

$\text{Volume } = \pi\int_a^b [\mathrm{f}(x) - k]^2\,\mathrm{d}x$</td><td>

$\text{Volume } = \pi\int_c^d [\mathrm{g}(y) - h]^2\,\mathrm{d}y$</td></tr>
</table>

B. Hollow Solid – 2 squares [$]^2$ – [$]^2$

Volume of a **hollow** solid of revolution about a coordinate axis	
 Volume $=\pi\int_a^b \{[f_1(x)]^2-[f_2(x)]^2\}dx$	 Volume $=\pi\int_a^b \{[g_1(y)]^2-[g_2(y)]^2\}dy$
Volume of a **hollow** solid of revolution about a line parallel to a coodinate axis	
 Volume $=\pi\int_a^b \{[f_1(x)-k]^2-[f_2(x)-k]^2\}dx$	 Volume $=\pi\int_c^d [g_1(y)-h]^2-[g_2(y)-h]^2\}dy$

Application of Definite Integration

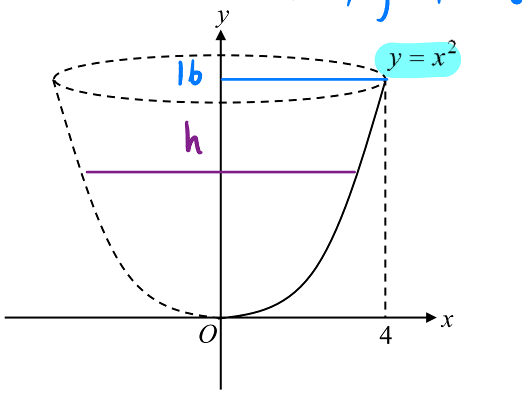
Example

30. The figure shows a bowl, which is formed by revolving part of the curve $y = x^2$ about the y-axis.
    (a) If the bowl holds water of height h units, express the volume of the water in the bowl in terms of h.
    (b) Find the capacity of the bowl.
    (c) Water is poured into the bowl at a rate of 2π cubic units/s, find the rate of change of the water level when $h = 8$.

$\frac{dh}{dt}$

$\frac{dV}{dt} = 2\pi$

$x=4,\ y = 4^2 = 16$



(a) volume

$= \int_0^h \pi x^2 \, dy$

$= \pi \int_0^h y \, dy$

$= \pi \left[\frac{1}{2} y^2 \right]_0^h$

$= \frac{1}{2}\pi h^2 = V$

(b) Capacity $= \frac{1}{2}\pi \cdot 16^2 = 128\pi$

(c) $V = \frac{1}{2}\pi h^2$

$\frac{dV}{dh} = \frac{1}{2}\pi \cdot 2h = \pi h.$

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$2\pi = \pi h \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{2}{h}$

$\therefore$ required rate of change

$= \left.\frac{dh}{dt}\right|_{h=8}$

$= \frac{2}{8}$

$= \frac{1}{4}$