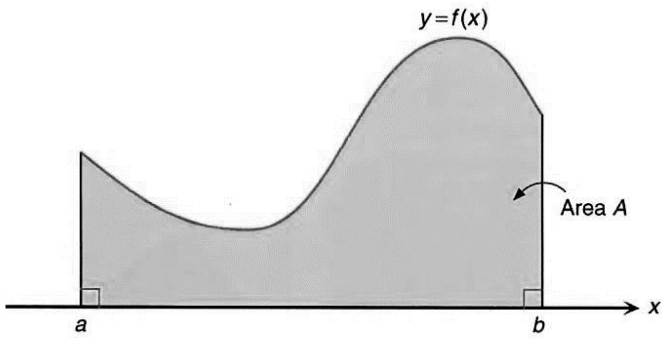
Chapter 8 Definite Integration
Supplementary Notes

Name: ________________________________ (   )   Class: F.5 _____

8.1 Concept of Definite Integration

A. Definition of Definite Integration

Aim: To find the area A of the region bounded by $y = f(x)$, the x-axis, the vertical lines $x = a$ and $x = b$, where $f(x)$ is a non-negative continuous function defined on the interval $[a,b]$.

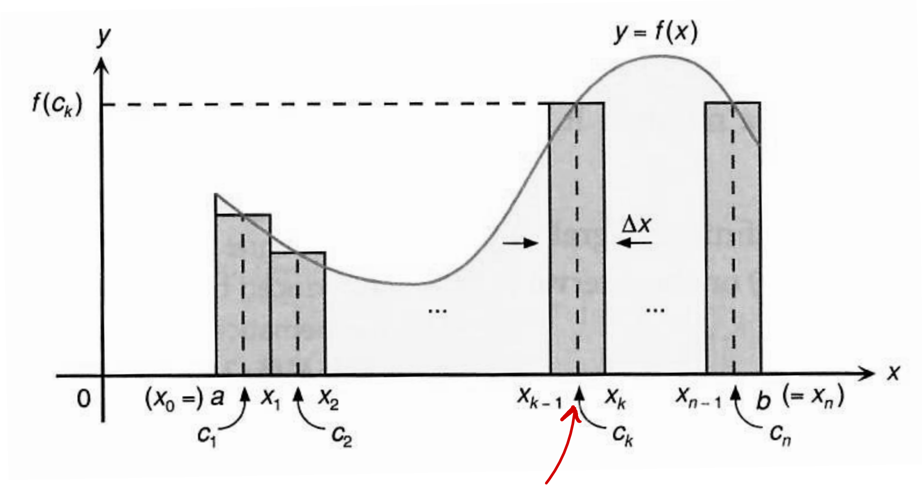


The following procedure shows the process of finding the area A.

1. Divide the interval $[a,b]$ into n equal sub-intervals by the points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$, where

   $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

   Hence, we divide the interval $[a,b]$ into n closed sub-intervals $[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n]$.



2. The width of each sub-interval is given by $\boxed{\Delta x = \dfrac{b-a}{n}.}$

3. Select an arbitrary point c_k in the kth sub-interval $[x_{k-1}, x_k]$ and consider the rectangle in the interval of height $f(c_k)$.

Area of the rectangle in the kth sub-interval $= f(c_k) \cdot \Delta x$

4. Sum of areas of all the rectangles under the curve $y = f(x)$ on $[a, b]$

$$= f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x$$

$$= \sum_{k=1}^{n} f(c_k)\Delta x$$

5. When the width Δx of each sub-interval tends to zero, the expression gives the area A of the region.

i.e. $A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(c_k)\Delta x$

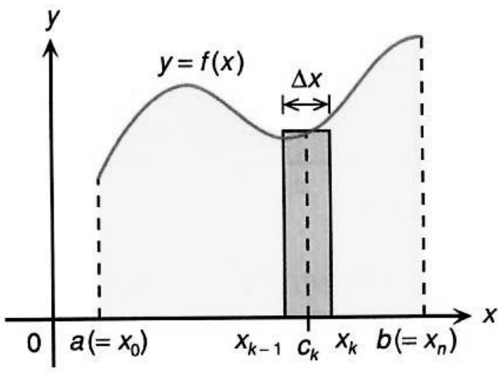
$n \to \infty$

This limit is, in fact, the **definite integral** of $f(x)$ from $x = a$ to $x = b$.

Definition of Definite Integral

For a continuous function $f(x)$ defined on the interval $[a, b]$, the definite integral of $f(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(c_k)\Delta x .$$



Note:

1. a is the lower limit of integral and b is the upper limit of integral.

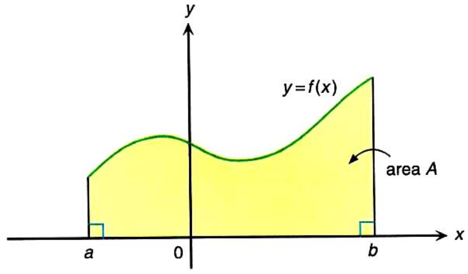
2. x is the variable of integration and the function $f(x)$ is the integrand.

3. The process of finding the definite integral of a function is called **definite integration**.

Unlike an indefinite integral, a definite integral is a **number**. It is not a function.

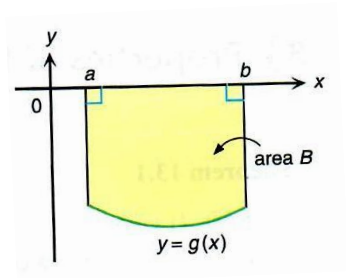
Take the following graph $y = \mathrm{f}(x)$ as an example.

$\int_a^b \mathrm{f}(x)\mathrm{d}x = A$



If $\mathrm{g}(x) < 0$ for all x in the interval $[a,b]$, and thus

$\int_a^b \mathrm{g}(x)\mathrm{d}x = -B$.



B. Properties of Definite Integrals

Definition:

For a continuous function $\mathrm{f}(x)$ on the interval $[a,b]$,

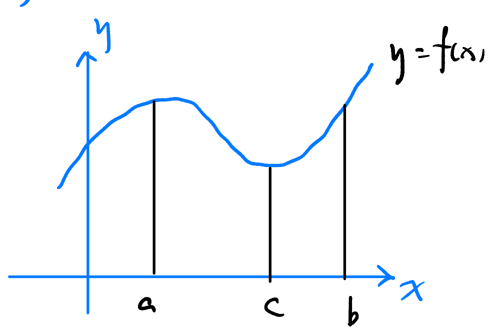
$$\int_a^b \mathrm{f}(x)\mathrm{d}x = -\int_b^a \mathrm{f}(x)\mathrm{d}x \, .$$

Property 1

Let $\mathrm{f}(x)$ and $\mathrm{g}(x)$ be continuous functions on the interval $[a,b]$ and k be a constant. We have the following properties of definite integrals.

(a) $\int_a^b k\mathrm{f}(x)\mathrm{d}x = k\int_a^b \mathrm{f}(x)\mathrm{d}x$

(d)



(b) $\int_a^b [\mathrm{f}(x) \pm \mathrm{g}(x)]\mathrm{d}x = \int_a^b \mathrm{f}(x)\mathrm{d}x \pm \int_a^b \mathrm{g}(x)\mathrm{d}x$

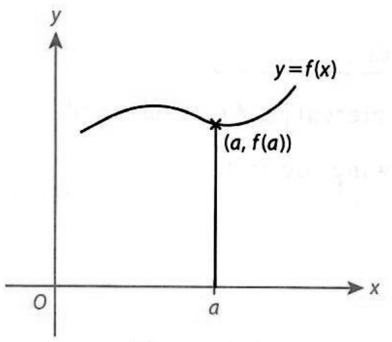
* (c) $\int_a^a \mathrm{f}(x)\mathrm{d}x = 0$ area under a point = 0

* (d) $\int_a^b \mathrm{f}(x)\mathrm{d}x = \int_a^c \mathrm{f}(x)\mathrm{d}x + \int_c^b \mathrm{f}(x)\mathrm{d}x$

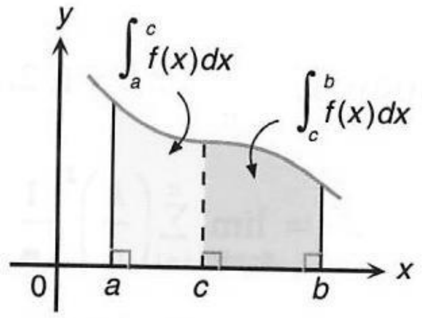
(e) $\int_a^b \mathrm{f}(x)\mathrm{d}x = \int_a^b \mathrm{f}(u)\mathrm{d}u$ x, u are dummy variables

Illustrations of Properties (c) to (e)

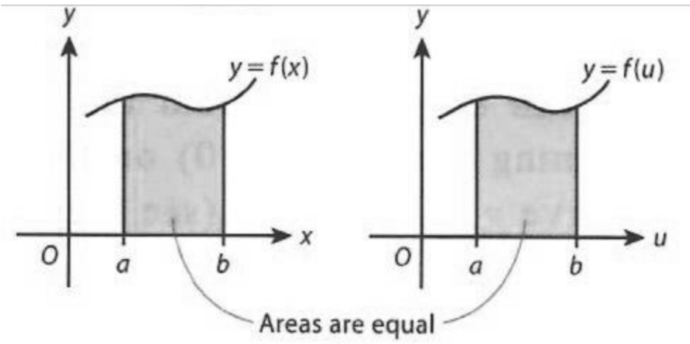
(c)



(d)



(e)



Example

1. Given that $\int_{-10}^{-5} f(x)dx = -6$ and $\int_{-5}^{1} f(x)dx = 10$, evaluate the following integrals.

(a) $\int_{-5}^{-5} f(x)dx = 0$

(b) $\int_{-5}^{-10} f(x)dx = -\int_{-10}^{-5} f(x)\,dx = -(-6) = 6$

(c) $\int_{-10}^{1} f(x)dx = \int_{-10}^{-5} f(x)\,dx + \int_{-5}^{1} f(x)\,dx$

$= -6 + 10$

$= 4$

C. Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on the interval $[a,b]$ and $F(x)$ is a primitive function of $f(x)$

(i.e. $\frac{d}{dx}[F(x)]=f(x)$), then $\int_a^b f(x)dx = F(b)-F(a)$.

Note: $F(b)-F(a)$ usually denoted as $[F(x)]_a^b$ or $F(x)\Big|_a^b$.

Example

2. Evaluate $\int_1^3 x\,dx$.

$\int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3 = \frac{3^2}{2} - \frac{1^2}{2} = 4$

3. Evaluate the following definite integrals.

(a) $\int_0^2 x^4 dx$

$= \left[\frac{x^5}{5}\right]_0^2$

$= \frac{32}{5}$

(b) $\int_4^{36} x^{-\frac{1}{2}} dx$

$= \left[2x^{\frac{1}{2}}\right]_4^{36}$

$= 2(6-2)$

$= 8$

(c) $\int_4^8 \frac{4}{x} dx$

$= [4\ln x]_4^8$

$= 4(\ln 8 - \ln 4)$

$= 4\ln 2$

(d) $\int_1^2 \frac{e^x + e^{-x}}{2} dx$

$= \frac{1}{2}\left[e^x - e^{-x}\right]_1^2$

$= \frac{1}{2}\left[e^2 - e^{-2} - (e^1 - e^{-1})\right]$

$= \frac{1}{2}\left(e^2 - \frac{1}{e^2} - e + \frac{1}{e}\right)$

(e) $\int_{-2}^{0}(x^4-2x^3-1)\mathrm{d}x$

$= \left[\frac{x^5}{5}-\frac{x^4}{2}-x\right]_{-2}^{0}$

$= 0-\left(\frac{-32}{5}-\frac{16}{2}+2\right)$

$= \frac{62}{5}$

(f) $\int_{4}^{9}(3x^{-2}+x^{\frac{1}{2}})\mathrm{d}x$

$= \left[-3x^{-1}+\frac{2x^{\frac{3}{2}}}{3}\right]_{4}^{9}$

$= -\frac{1}{3}+18-\left(\frac{-3}{4}+\frac{16}{3}\right)$

$= \frac{157}{12}$

(g) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 3\sin x \, \mathrm{d}x$

$= \left[-3\cos x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= -3\cos\frac{\pi}{2}-\left(-3\cos\frac{\pi}{3}\right)$

$= \frac{3}{2}$

(h) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\cos x \, \mathrm{d}x$

$= \left[2\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= 2\sin\frac{\pi}{4}-2\sin\frac{\pi}{6}$

$= \sqrt{2}-1$

(i) $\int_{0}^{\frac{\pi}{4}} \tan^2 x\mathrm{d}x$

$= \int_{0}^{\frac{\pi}{4}}(\sec^2 x-1)\,dx$

$= \left[\tan x - x\right]_{0}^{\frac{\pi}{4}}$

$= \tan\frac{\pi}{4}-\frac{\pi}{4}$

$= 1-\frac{\pi}{4}$

(j) $\int_{0}^{\frac{\pi}{2}} \sin 2x \, \mathrm{d}x$

$= \left[\frac{-\cos 2x}{2}\right]_{0}^{\frac{\pi}{2}}$

$= -\frac{1}{2}(\cos\pi-\cos 0)$

$= 1$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

(k) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\tan x \sin 2x}{\cos^2 x}\,dx$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^3 x} \cdot 2\sin x \cos x\,dx$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\tan^2 x\,dx$$

$$= 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\sec^2 x - 1)\,dx$$

$$= 2\left[\tan x - x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= 2\left[\sqrt{3} - \frac{\pi}{3} - \left(-\sqrt{3} + \frac{\pi}{3}\right)\right]$$

$$= 4\sqrt{3} - \frac{4\pi}{3}$$

(l) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1-\cos 2x}{1+\cos x}\,dx$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2\sin^2 x}{1+\cos x}\,dx$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1-\cos^2 x}{1+\cos x}\,dx$$

$(1+\cos x)(1-\cos x)$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1-\cos x)\,dx$$

$$= 2\left[x - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\left[\frac{\pi}{2} - 1 - \left(\frac{\pi}{6} - \frac{1}{2}\right)\right]$$

$$= \frac{2\pi}{3} - 1$$

4. Given $\displaystyle\int_1^3 \frac{g(x)}{g(x)+h(x)}\,dx = 4$, evaluate the following integrals.

(a) $\displaystyle\int_3^1 \frac{g(t)}{g(t)+h(t)}\,dt$

$$= -\int_1^3 \frac{g(x)}{g(x)+h(x)}\,dx$$

$$= -4$$

(b) $\displaystyle\int_3^1 \frac{h(t)}{g(t)+h(t)}\,dt$

$$= -\int_1^3 \frac{h(x)}{g(x)+h(x)}\,dx$$

$$= -\int_1^3 \frac{g(x)+h(x)-g(x)}{g(x)+h(x)}\,dx$$

$$= -\left[\int_1^3 \left(1 - \frac{g(x)}{g(x)+h(x)}\right)dx\right]$$

$$= -\int_1^3 1\,dx + \int_1^3 \frac{g(x)}{g(x)+h(x)}\,dx$$

$$= -(3-1) + 4$$

$$= 2$$

5. (a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{4}x^2(\ln x^2 - 1)\right]$.

(b) Hence evaluate $\int_1^e 4x\ln x\,\mathrm{d}x$.

(a) $\frac{d}{dx}\left[\frac{1}{4}x^2(2\ln x - 1)\right]$

$= \frac{1}{4}\left[x^2 \cdot \frac{2}{x} + (2\ln x - 1)\cdot 2x\right]$

$= \frac{1}{4}(2x + 4x\ln x - 2x)$

$= x\ln x$

(b) $\int_1^e 4x\ln x\,dx$

$= 4\int_1^e \frac{d}{dx}\left[\frac{1}{4}x^2(\ln x^2 - 1)\right]dx$

$= \int_1^e \frac{d}{dx}\left[x^2(\ln x^2 - 1)\right]dx.$

$= \left[x^2(\ln x^2 - 1)\right]_1^e$

$= e^2(\ln e^2 - 1) - 1^2(\ln 1^2 - 1)$

$= e^2 + 1$

6. (a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x)$.

(b) Hence evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\mathrm{d}x$.

(a) $\frac{d}{dx}(x\sin x)$

$= x\cos x + \sin x$

(b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{d}{dx}(x\sin x) - \sin x\right)dx$

$= \left[x\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx$

$= \frac{\pi}{2}\sin\frac{\pi}{2} - \left(-\frac{\pi}{2}\sin\left(-\frac{\pi}{2}\right)\right) - \left[-\cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 0$

8.2 Integration by Substitution

Theorem

If $u = \mathrm{g}(x)$ is a differentiable function of x, then

$$\int_a^b \mathrm{f}(\mathrm{g}(x))\mathrm{g}'(x)\mathrm{d}x = \int_{\mathrm{g}(a)}^{\mathrm{g}(b)} \mathrm{f}(u)\mathrm{d}u .$$

Example

*Change **the lower limit** and **the upper limit.***

7. Evaluate $\int_{-1}^{3} (3x-4)^3 \mathrm{d}x$.

Let $u =$ $3x-4$, then $\mathrm{d}u =$ $3dx$

When $x=-1$, $u =$ -7 ; When $x = 3$, $u =$ 5 .

$\int_{-7}^{5} u^3 \cdot \frac{1}{3} du$

$= \left[\frac{u^4}{12} \right]_{-7}^{5}$

$= -148$

OR

$\int_{-1}^{3} \frac{1}{3} (3x-4)^3 d(3x-4)$

$= \left[\frac{(3x-4)^4}{12} \right]_{-1}^{3}$

$= -148$

8. Evaluate the following definite integrals.

(a) $\int_2^6 \sqrt{4x+1}\, \mathrm{d}x$

Let $u = 4x+1$

$du = 4dx$

when $x=2$, $u=9$

$x=6$, $u=25$

$= \int_9^{25} \sqrt{u} \cdot \frac{1}{4} du$

$= \frac{1}{4} \left[\frac{2u^{\frac{3}{2}}}{3} \right]_9^{25}$

$= \frac{1}{4} \left(\frac{250}{3} - 18 \right)$

$= \frac{49}{3}$

(b) $\int_{-1}^{2} 4x\sqrt{2x^2+1}\,dx$

let $u = 2x^2+1$

$du = 4x\,dx$

when $x=-1$, $u=3$

$x=2$, $u=9$

$= \int_3^9 \sqrt{u}\,du$

$= \left[\frac{2u^{\frac{3}{2}}}{3}\right]_3^9$

$= \frac{2}{3}(27 - \sqrt{27})$

$= \frac{2}{3}(27 - 3\sqrt{3})$

$= 18 - 2\sqrt{3}$

(c) $\int_{-3}^{2} \frac{x}{\sqrt{6-x}}\,dx$

let $u = 6-x$, $x = 6-u$

$du = -dx$

when $x=-3$, $u=9$

$x=2$, $u=4$

$= \int_9^4 \frac{6-u}{\sqrt{u}}(-du)$

$\int_a^b = -\int_b^a$

$= \int_4^9 (6u^{-\frac{1}{2}} - u^{\frac{1}{2}})\,du$

$= \left[\frac{6u^{\frac{1}{2}}}{(\frac{1}{2})} - \frac{2u^{\frac{3}{2}}}{3}\right]_4^9$

$= 36 - 18 - (24 - \frac{16}{3})$

$= -\frac{2}{3}$

(d) $\int_{-1}^{0} (x+1)(x^2+2x+5)^4 \, dx$

$= \int_{-1}^{0} \frac{1}{2}(x^2+2x+5)^4 \, d(x^2+2x+5)$

$= \frac{1}{2}\left[\frac{(x^2+2x+5)^5}{5}\right]_{-1}^{0}$

$= \frac{1}{10}(5^5-4^5)$

$= \frac{2101}{10}$

(e) $\int_{-2}^{-1} \frac{x}{(x^2+3)^2} \, dx$

$= \frac{1}{2}\int_{-2}^{-1} \frac{1}{(x^2+3)^2} \, d(x^2+3)$

$= \frac{1}{2}\left[\frac{(x^2+3)^{-1}}{-1}\right]_{-2}^{-1}$

$= -\frac{1}{2}\left(\frac{1}{4}-\frac{1}{7}\right)$

$= \frac{-3}{56}$

$(\cos x)^{-1} \neq \cos^{-1} x$

(f) $\int_{-\frac{\pi}{3}}^{0} \frac{\sin x}{\cos^2 x} \mathrm{d}x$

$= -\int_{-\frac{\pi}{3}}^{0} \frac{1}{\cos^2 x} \, d\cos x$

$= -\left[\frac{(\cos x)^{-1}}{-1} \right]_{-\frac{\pi}{3}}^{0}$

$= \frac{1}{\cos 0} - \frac{1}{\cos(-\frac{\pi}{3})}$

$= -1$

$\int_{-\frac{\pi}{3}}^{0} \frac{\sin x}{\cos^2 x} dx$

$= \int_{-\frac{\pi}{3}}^{0} \tan x \sec x \, dx$

$= [\sec x]_{-\frac{\pi}{3}}^{0}$

(g) $\int_{0}^{\frac{\pi}{2}} 8\cos^2 x \mathrm{d}x$

$= \int_{0}^{\frac{\pi}{2}} 8 \left(\frac{1+\cos 2x}{2} \right) dx$

$= 4 \left[x + \frac{\sin 2x}{2} \right]_{0}^{\frac{\pi}{2}}$

$= 4 \left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right)$

$= 2\pi$

9. Evaluate the following definite integrals.

(a) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

$x^2 = 4\sin^2\theta$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

when $x = 0$,

$\sin\theta = 0$, $\theta = 0$

when $x = 1$

$\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

$= \int_0^{\frac{\pi}{6}} \frac{1}{2\cos\theta} \cdot 2\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{6}} 1\, d\theta$

$= [\theta]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{6}$.

	Substitution
$\frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$ ✓
$\sqrt{a^2-x^2}$	or $x = a\cos\theta$
$\frac{1}{x^2+a^2}$	$x = a\tan\theta$ ✓

Principal Value

$\sin\theta \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ✓

$\cos\theta \to [0, \pi]$

$\tan\theta \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ✓

e.g. $\sin\theta = -\frac{1}{2}$

$\theta = -\frac{\pi}{6}$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}$

$\tan\theta = -1$

$\theta = -\frac{\pi}{4}$

(b) $\int_0^3 \frac{dx}{x^2+9}$

Let $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

when $x = 0$,

$\tan\theta = 0$, $\theta = 0$

when $x = 3$

$1 = \tan\theta$

$\theta = \frac{\pi}{4}$

$= \int_0^{\frac{\pi}{4}} \frac{1}{9(1+\tan^2\theta)} \cdot 3\sec^2\theta\, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{1}{3}\, d\theta$

$= \left[\frac{1}{3}\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{12}$

(c) $\int_{\frac{3}{4}}^{\frac{3}{2}} \sqrt{9-4x^2}\,dx$

$4x^2 = 9\sin^2\theta$

Let $2x = 3\sin\theta$

$x = \frac{3}{2}\sin\theta$

$dx = \frac{3}{2}\cos\theta\, d\theta$

when $x = \frac{3}{4}$

$\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

when $x = \frac{3}{2}$

$\sin\theta = 1$, $\theta = \frac{\pi}{2}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta} \cdot \frac{3}{2}\cos\theta\, d\theta$$

$$= \frac{3}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\cos\theta \cdot \cos\theta\, d\theta$$

$$= \frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{9}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{9}{4}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - \left(\frac{\pi}{6} + \frac{\sin\frac{\pi}{3}}{2}\right)\right]$$

$$= \frac{9}{4}\left(\frac{\pi}{2} - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{3\pi}{4} - \frac{9\sqrt{3}}{16}$$

(d) $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{4\sin^2\theta\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{4\sin^2\theta\ 2\cos\theta} \cdot 2\cos\theta\, d\theta$$

$$= \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc^2\theta\, d\theta$$

$$= \frac{1}{4}\left[-\cot\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{4}\left[\frac{-1}{\tan\theta}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{4}\left(-\frac{1}{\sqrt{3}}+1\right)$$

$$= \frac{1}{4}\left(-\frac{\sqrt{3}}{3}+1\right)$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{12}$$

$x^2 = 4\sin^2\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

when $x = \sqrt{2}$

$\sin\theta = \frac{\sqrt{2}}{2}$, $\theta = \frac{\pi}{4}$

when $x = \sqrt{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$, $\theta = \frac{\pi}{3}$

$$\boxed{\begin{aligned} d\cot\theta &= -\csc^2\theta \\ d\sec\theta &= \sec\theta\tan\theta \\ d\csc\theta &= -\csc\theta\cot\theta \end{aligned}}$$

(d) $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$

Let $x = 2\cos\theta$

$dx = -2\sin\theta\, d\theta$

when $x = \sqrt{2}$,

$\cos\theta = \frac{\sqrt{2}}{2}$, $\theta = \frac{\pi}{4}$

when $x = \sqrt{3}$

$\cos\theta = \frac{\sqrt{3}}{2}$, $\theta = \frac{\pi}{6}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{1}{4\cos^2\theta\sqrt{4-4\cos^2\theta}}(-2\sin\theta)\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{4\cos^2\theta \cdot 2\sin\theta} \cdot 2\sin\theta\,d\theta$$

$$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2\theta\,d\theta$$

$$= \frac{1}{4}\left[\tan\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left(\tan\frac{\pi}{4} - \tan\frac{\pi}{6}\right)$$

$$= \frac{1}{4}\left(1 - \frac{\sqrt{3}}{3}\right)$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{12}$$

10. (a) Show that $\frac{2}{(t+1)(t+3)} \equiv \frac{1}{t+1} - \frac{1}{t+3}$.

(b) Using the substitution $t = \frac{1}{x}$, evaluate $\int_1^2 \frac{1}{(1+x)(1+3x)}\,dx$.

(a) $\text{R.H.S.} = \frac{(t+3)-(t+1)}{(t+1)(t+3)} = \frac{t+3-t-1}{(t+1)(t+3)} = \frac{2}{(t+1)(t+3)}$

(b) $\int_1^2 \frac{1}{(1+x)(1+3x)}\,dx$

Let $t = \frac{1}{x}$

$dt = -x^{-2}\,dx$

$dt = \frac{-1}{x^2}\,dx$

$dt = -t^2\,dx$

$-\frac{1}{t^2}\,dt = dx$

When $x=1$, $t=1$

When $x=2$, $t=\frac{1}{2}$

$= \int_1^{\frac{1}{2}} \frac{1}{(1+\frac{1}{t})(1+\frac{3}{t})}\left(-\frac{1}{t^2}\,dt\right)$

$= \int_{\frac{1}{2}}^{1} \frac{t^2}{(t+1)(t+3)} \cdot \frac{1}{t^2}\,dt$

$= \int_{\frac{1}{2}}^{1} \frac{1}{(t+1)(t+3)}\,dt$

$= \frac{1}{2}\int_{\frac{1}{2}}^{1} \left(\frac{1}{t+1} - \frac{1}{t+3}\right) dt$

$= \frac{1}{2}\left[\ln(t+1) - \ln(t+3)\right]_{\frac{1}{2}}^{1}$

$= \frac{1}{2}\left(\ln 2 - \ln 4 - \ln\frac{3}{2} + \ln\frac{7}{2}\right)$

$= \frac{1}{2}\ln\frac{7}{6}$

11. (a) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, where a is a constant.

(b) Hence, show that $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$.

$a = ?$

$f(x) = ?$

(c) Using the result of (b), evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$.

(a) $\int_0^a f(x)\,dx$

Let $x = a - u$

$dx = -du$

when $x = 0$, $u = a$

$x = a$, $u = 0$

$= \int_a^0 f(a-u)(-du)$

$= \int_0^a f(a-u)\,du$

x, u - dummy variable

$= \int_0^a f(a-x)\,dx$

(b) Put $a = \frac{\pi}{2}$, $f(x) = \frac{\sin x}{\sin x + \cos x}$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx$$

(c) $$2\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} 1\,dx \leftarrow [x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$$

12. (a) Show that $\int_0^{\frac{a}{2}} \mathrm{f}(a-x)\mathrm{d}x = \int_{\frac{a}{2}}^{a} \mathrm{f}(x)\mathrm{d}x$, where a is a constant.

Hence show that $\int_0^{a} \mathrm{f}(x)\mathrm{d}x = \int_0^{\frac{a}{2}} [\mathrm{f}(x)+\mathrm{f}(a-x)]\mathrm{d}x$.

$\int_0^a = \int_0^{\frac{a}{2}} + \int_{\frac{a}{2}}^a$

(b) Hence evaluate $\int_0^{\pi} \left(e^{\left(x-\frac{\pi}{2}\right)^2} \cos^3 x + x \right) \mathrm{d}x$.

(a) $\int_0^{\frac{a}{2}} f(a-x)\,dx$

Let $a - x = u$

$-dx = du$

when $x = 0$, $u = a$

$x = \frac{a}{2}$, $u = \frac{a}{2}$

$= \int_a^{\frac{a}{2}} f(u)\,(-du)$

$= \int_{\frac{a}{2}}^{a} f(u)\,du$

$= \int_{\frac{a}{2}}^{a} f(x)\,dx$

$\int_0^a f(x)\,dx = \int_0^{\frac{a}{2}} f(x)\,dx + \int_{\frac{a}{2}}^{a} f(x)\,dx$

$= \boxed{\int_0^{\frac{a}{2}}} f(x)\,dx + \boxed{\int_0^{\frac{a}{2}}} f(a-x)\,dx$

$= \int_0^{\frac{a}{2}} [f(x) + f(a-x)]\,dx$

12. (a) Show that $\int_0^{\frac{a}{2}} \mathrm{f}(a-x)\mathrm{d}x = \int_{\frac{a}{2}}^{a} \mathrm{f}(x)\mathrm{d}x$, where a is a constant.

Hence show that $\int_0^{a} \mathrm{f}(x)\mathrm{d}x = \int_0^{\frac{a}{2}} [\mathrm{f}(x)+\mathrm{f}(a-x)]\mathrm{d}x$.

(b) Hence evaluate $\int_0^{\pi} \left(e^{\left(x-\frac{\pi}{2}\right)^2} \cos^3 x + x \right) \mathrm{d}x$.

$\cos(\pi - x)$
$= -\cos x$

(b) Put $f(x) = e^{(x-\frac{\pi}{2})^2} \cos^3 x$, $a = \pi$

$f(\pi - x) = e^{(\pi - x - \frac{\pi}{2})^2} \cos^3(\pi - x)$

$= e^{(\frac{\pi}{2} - x)^2} (-\cos x)^3$

$= e^{(x-\frac{\pi}{2})^2} (-\cos^3 x)$

$= -e^{(x-\frac{\pi}{2})^2} \cos^3 x$

$\therefore f(x) + f(\pi - x) = 0$

$\therefore \int_0^{\pi} e^{(x-\frac{\pi}{2})^2} \cos^3 x \, dx$

$= \int_0^{\frac{\pi}{2}} 0 \, dx$

$= 0$

$\therefore$ required result $= \int_0^{\pi} x \, dx$

$= \left[\frac{x^2}{2} \right]_0^{\pi}$

$= \frac{\pi^2}{2}$

13. (a) Let $f(x)$ be a function such that $f(x) = f(a-x)$ for all x. , a is a constant.

Prove that $\int_0^a x f(x)\, dx = \frac{a}{2}\int_0^a f(x)\, dx$.

(b) Hence find $\int_0^{\pi} x \sin x \cos^4 x\, dx$.

(a) $\int_0^a x f(x)\, dx$

let $x = a-u$

$dx = -du$

when $x = 0$, $u = a$

$x = a$, $u = 0$

$= \int_a^0 (a-u) f(a-u)(-du)$

$= \int_0^a (a-u) f(u)\, du$

$= \int_0^a (a-x) f(x)\, dx$

$= \int_0^a a f(x)\, dx - \int_0^a x f(x)\, dx$

$\therefore 2\int_0^a x f(x)\, dx = \int_0^a a f(x)\, dx$

$\int_0^a x f(x)\, dx = \frac{a}{2}\int_0^a f(x)\, dx$

Condition.

13. (a) Let $f(x)$ be a function such that $f(x)=f(a-x)$ for all x.

Prove that $\int_0^a xf(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$.

$a=\pi$

$f(x) = \sin x \cos^4 x$

(b) Hence find $\int_0^\pi x\sin x\cos^4 x\,dx$.

(b) Put $f(x) = \sin x \cos^4 x$, $a=\pi$

$$\left.\begin{aligned} f(\pi-x) &= \sin(\pi-x)\cos^4(\pi-x) \\ &= \sin x\cdot(-\cos x)^4 \leftarrow \text{key step.} \\ &= \sin x\cos^4 x \end{aligned}\right\} \text{Check condition 有冇}$$

$$= f(x)$$

$$\int_0^\pi x\sin x\cos^4 x\,dx = \frac{\pi}{2}\int_0^\pi \sin x\cos^4 x\,dx$$

$$= -\frac{\pi}{2}\int_0^\pi \cos^4 x\,d\cos x$$

$$= -\frac{\pi}{2}\left[\frac{\cos^5 x}{5}\right]_0^\pi$$

$$= -\frac{\pi}{2}\left(-\frac{1}{5}-\frac{1}{5}\right)$$

$$= \frac{\pi}{5}$$

$\int_0^a \to \int_0^a$ Let $x = a-u$

14. Let $f(x)$ be a continuous function.

(a) Show that $\int_0^{2\pi} f(x)dx = \int_0^{2\pi} f(2\pi - x)dx$.

condition.

(b) Further, if $f(x)+f(2\pi-x)=k$ for all x, show that $\int_0^{2\pi} f(x)dx = k\pi$.

Hence, evaluate $\int_0^{2\pi} \frac{1}{1+e^{2\sin x}}dx$.

Let $x = 2\pi - u$, $dx = -du$

when $x=0$, $u=2\pi$

$x=2\pi$, $u=0$

(a)
$$\int_0^{2\pi} f(x)\,dx$$
$$= \int_{2\pi}^{0} f(2\pi-u)(-du)$$
$$= \int_0^{2\pi} f(2\pi-u)\,du$$
$$= \int_0^{2\pi} f(2\pi-x)\,dx$$

(b)
$$\int_0^{2\pi} [f(x)+f(2\pi-x)]\,dx = \int_0^{2\pi} k\,dx$$
$$\int_0^{2\pi} f(x)\,dx + \int_0^{2\pi} f(2\pi-x)\,dx = [kx]_0^{2\pi}$$
$$2\int_0^{2\pi} f(x)\,dx = k(2\pi-0)$$
$$\int_0^{2\pi} f(x)\,dx = k\pi$$

Let $f(x) = \frac{1}{1+e^{2\sin x}}$

$\sin(2\pi - x) = -\sin x$

$$f(x)+f(2\pi-x) = \frac{1}{1+e^{2\sin x}} + \frac{1}{1+e^{2\sin(2\pi-x)}}$$
$$= \frac{1}{1+e^{2\sin x}} + \frac{1}{1+e^{-2\sin x}}$$
$$= \frac{1}{1+e^{2\sin x}} + \frac{1}{1+\frac{1}{e^{2\sin x}}}$$
$$= \frac{1}{1+e^{2\sin x}} + \frac{e^{2\sin x}}{1+e^{2\sin x}}$$
$$= 1$$

$$\frac{1}{\left(\frac{e^{2\sin x}+1}{e^{2\sin x}}\right)}$$

$$\therefore \int_0^{2\pi} \frac{1}{1+e^{2\sin x}}\,dx = \pi$$

15. (a) (i) Evaluate $\int_0^{\pi} \cos^2 x \, dx$.

(ii) Using the substitution $x = \pi - y$, evaluate $\int_0^{\pi} x\cos^2 x \, dx$.

(b) (i) Show that $\int_{\pi}^{2\pi} x\cos^2 x \, dx = \pi \int_0^{\pi} \cos^2 x \, dx + \int_0^{\pi} x\cos^2 x \, dx$.

(ii) Show that $\int_0^{2\pi} x\cos^2 x \, dx = \pi^2$.

(c) Using the result of (b)(ii), evaluate $\int_0^{\sqrt{2\pi}} x^3 \cos^2 x^2 \, dx$.

$u = x - \pi$

when $x = \pi$, $u = 0$

$x = 2\pi$, $u = \pi$

$\int_0^{2\pi} \rightarrow \int_0^{\pi} + \int_{\pi}^{2\pi}$

$x^2 = u$, $2x = \frac{du}{dx}$

$2x\,dx = du$

when $x = 0$, $u = 0$

$x = \sqrt{2\pi}$, $u = 2\pi$

(a) (i) $\int_0^{\pi} \cos^2 x \, dx$

$= \int_0^{\pi} \frac{1}{2}(1 + \cos 2x) \, dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$

$= \frac{\pi}{2}$

(ii) $\int_0^{\pi} x \cos^2 x \, dx$

Let $x = \pi - y$

$dx = -dy$

when $x = 0$, $y = \pi$

$x = \pi$, $y = 0$

$= \int_{\pi}^{0} (\pi - y)\cos^2(\pi - y)(-dy)$

$= \int_0^{\pi} (\pi - x)\cos^2 x \, dx$

$= \pi \int_0^{\pi} \cos^2 x \, dx - \int_0^{\pi} x\cos^2 x \, dx$

$\therefore 2\int_0^{\pi} x\cos^2 x \, dx = \frac{\pi^2}{2}$

$\int_0^{\pi} x\cos^2 x \, dx = \frac{\pi^2}{4}$

15. (a) (i) Evaluate $\int_0^{\pi} \cos^2 x \, dx$.

(ii) Using the substitution $x = \pi - y$, evaluate $\int_0^{\pi} x \cos^2 x \, dx$.

(b) (i) Show that $\int_{\pi}^{2\pi} x \cos^2 x \, dx = \pi \int_0^{\pi} \cos^2 x \, dx + \int_0^{\pi} x \cos^2 x \, dx$.

(ii) Show that $\int_0^{2\pi} x \cos^2 x \, dx = \pi^2$.

(c) Using the result of (b)(ii), evaluate $\int_0^{\sqrt{2\pi}} x^3 \cos^2 x^2 \, dx$.

(b) (i) $\int_{\pi}^{2\pi} x \cos^2 x \, dx$

Let $u = x - \pi$

$du = dx$

when $x = \pi$, $u = 0$

$x = 2\pi$, $u = \pi$

$= \int_0^{\pi} (u+\pi) \cos^2 (u+\pi) \, du$

$= \int_0^{\pi} u \cos^2 u \, du + \int_0^{\pi} \pi \cos^2 u \, du$

$= \pi \int_0^{\pi} \cos^2 x \, dx + \int_0^{\pi} x \cos^2 x \, dx$

(ii) $\int_0^{2\pi} x \cos^2 dx$

$= \int_0^{\pi} x \cos^2 x \, dx + \int_{\pi}^{2\pi} x \cos^2 x \, dx$

$= \frac{\pi^2}{4} + \pi \cdot \int_0^{\pi} \cos^2 x \, dx + \int_0^{\pi} x \cos^2 x \, dx$

$= \frac{\pi^2}{4} + \pi \cdot \frac{\pi}{2} + \frac{\pi^2}{4}$

$= \pi^2$

15. (a) (i) Evaluate $\int_0^{\pi} \cos^2 x \, dx$.

(ii) Using the substitution $x = \pi - y$, evaluate $\int_0^{\pi} x\cos^2 x \, dx$.

(b) (i) Show that $\int_{\pi}^{2\pi} x\cos^2 x \, dx = \pi\int_0^{\pi} \cos^2 x \, dx + \int_0^{\pi} x\cos^2 x \, dx$.

(ii) Show that $\int_0^{2\pi} x\cos^2 x \, dx = \pi^2$.

(c) Using the result of (b)(ii), evaluate $\int_0^{\sqrt{2\pi}} x^3 \cos^2 x^2 \, dx$.

(c) $\int_0^{\sqrt{2\pi}} x^3 \cos^2 x^2 \, dx$

$= \int_0^{2\pi} \underset{\uparrow\, x^2}{\underline{u}} \cos^2 u \cdot \underset{\uparrow\, x\,dx}{\underline{\tfrac{1}{2} du}}$

$= \frac{1}{2} \cdot \int_0^{2\pi} x \cos^2 x \, dx$

$= \frac{\pi^2}{2}$

Let $x^2 = u$

$2x = \frac{du}{dx}$

$2x\,dx = du$

when $x = 0$, $u = 0$

$x = \sqrt{2\pi}$, $u = 2\pi$

8.3 Integration by Parts

Theorem

If u and v are differentiable functions of x, then

number.

$$\int_a^b u\mathrm{d}v = \boxed{[uv]_a^b} - \int_a^b v\mathrm{d}u .$$

Priority : ① e function

② sin / cos x

③ x^n

* ln x ← 留低

Example

16. Evaluate the following definite integrals.

(a) $\int_0^4 xe^x \mathrm{d}x$

$= \int_0^4 x\, de^x$

$= [xe^x]_0^4 - \int_0^4 e^x dx$

$= 4 \cdot e^4 - 0 \cdot e^0 - [e^x]_0^4$

$= 4e^4 - (e^4 - e^0)$

$= 3e^4 + 1$

(b) $\int_0^2 (x+1)e^x \mathrm{d}x$

$= \int_0^2 (x+1)\, de^x$

$= [(x+1)e^x]_0^2 - \int_0^2 e^x \cdot dx$

$= 3e^2 - 1 - [e^x]_0^2$

$= 3e^2 - 1 - (e^2 - 1)$

$= 2e^2$

(c) $\int_1^{e^2} x^3 \ln x \, dx$

$\ln e^2 = 2 \ , \ \ln 1 = 0$

$= \frac{1}{4}\int_1^{e^2} \ln x \, dx^4$

$= \frac{1}{4}\left\{ \left[x^4 \ln x\right]_1^{e^2} - \int_1^{e^2} x^4 \cdot \frac{1}{x} dx \right\}$

$= \frac{1}{4}\left(2e^8 - \int_1^{e^2} x^3 dx\right)$

$= \frac{1}{4}\left(2e^8 - \left[\frac{x^4}{4}\right]_1^{e^2}\right)$

$= \frac{1}{4}\left[2e^8 - \left(\frac{1}{4}e^8 - \frac{1}{4}\right)\right]$

$= \frac{7}{16}e^8 + \frac{1}{16}$

(d) $\int_1^4 x \ln \frac{1}{x} dx$

$\ln \frac{1}{x} = \ln x^{-1} = -\ln x$

$= -\int_1^4 x \ln x \, dx$

$= -\frac{1}{2}\int_1^4 \ln x \, dx^2$

$= -\frac{1}{2}\left\{ \left[x^2 \ln x\right]_1^4 - \int_1^4 x^2 \cdot \frac{1}{x} dx \right\}$

$= -\frac{1}{2}\left(16 \ln 4 - \int_1^4 x \, dx\right)$

$= -\frac{1}{2}\left(16 \ln 4 - \left[\frac{x^2}{2}\right]_1^4\right)$

$= -\frac{1}{2}\left[16 \ln 4 - \left(8 - \frac{1}{2}\right)\right]$

$= \frac{15}{4} - 8 \ln 4$

(e) $\int_0^{\frac{\pi}{2}} x\sin x\,dx$

$= -\int_0^{\frac{\pi}{2}} x\,d\cos x$

$= -[x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,dx$

$= [\sin x]_0^{\frac{\pi}{2}}$

$= 1$

(f) $\int_0^{\frac{\pi}{2}} x\cos x\,dx$

$= \int_0^{\frac{\pi}{2}} x\,d\sin x$

$= [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx$

$= \frac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + 0 - 1$

$= \frac{\pi}{2} - 1$

(g) $\int_0^{\frac{\pi}{2}} x^2 \sin 3x dx$ 2次 by parts

$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} x^2 \, d\cos 3x$

$= -\frac{1}{3}\left[x^2\cos 3x\right]_0^{\frac{\pi}{2}} + \frac{1}{3}\int_0^{\frac{\pi}{2}} \cos 3x \cdot 2x \cdot dx$

$= \frac{2}{3}\int_0^{\frac{\pi}{2}} x\cos 3x \, dx$

$= \frac{2}{3}\cdot\frac{1}{3}\cdot\int_0^{\frac{\pi}{2}} x \, d\sin 3x$

$= \frac{2}{9}\left(\left[x\sin 3x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin 3x \, dx\right)$

$= \frac{2}{9}\left(-\frac{\pi}{2} + \left[\frac{\cos 3x}{3}\right]_0^{\frac{\pi}{2}}\right)$

$= \frac{2}{9}\left(-\frac{\pi}{2} - \frac{1}{3}\right)$

$= -\frac{\pi}{9} - \frac{2}{27}$

(h) $\int_0^1 x^2 e^{-x}\,dx$

$= -\int_0^1 x^2\,d e^{-x}$

$= -\left[x^2 e^{-x}\right]_0^1 + \int_0^1 e^{-x} \cdot 2x\,dx$

$= -e^{-1} + 2\int_0^1 x e^{-x}\,dx$

$= -e^{-1} - 2\int_0^1 x\,de^{-x}$

$= -e^{-1} - 2\left(\left[xe^{-x}\right]_0^1 - \int_0^1 e^{-x}\,dx\right)$

$= -e^{-1} - 2\left(e^{-1} + \left[e^{-x}\right]_0^1\right)$

$= -e^{-1} - 2\left(e^{-1} + e^{-1} - 1\right)$

$= 2 - 5e^{-1}$

$= 2 - \frac{5}{e}$

17. Evaluate the following integrals.

(a) $\int_0^{\frac{\pi}{2}} e^{-x}\cos x dx$

2次 by parts. 變返 original

$\cos x \to -\sin x \to -\cos x$

$= -\int_0^{\frac{\pi}{2}} \cos x \, de^{-x}$

$= -\left[e^{-x}\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^{-x}\cdot(-\sin x)\,dx$

$= -(0-1) - \int_0^{\frac{\pi}{2}} e^{-x}\sin x dx$

$= 1 + \int_0^{\frac{\pi}{2}} \sin x \, de^{-x}$

$= 1 + \left[e^{-x}\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^{-x}\cos x\,dx$

$= 1 + e^{-\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^{-x}\cos x\,dx$

$\therefore \int_0^{\frac{\pi}{2}} e^{-x}\cos x\,dx = \frac{1+e^{-\frac{\pi}{2}}}{2}$

(b) $\int_0^{\frac{\pi}{2}} e^x \sin 2x dx$

$$= \int_0^{\frac{\pi}{2}} \sin 2x \, de^x$$

$$= \left[e^x \sin 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cdot \cos 2x \cdot 2 \, dx$$

$$= 0 - 2\int_0^{\frac{\pi}{2}} \cos 2x \, de^x$$

$$= -2 \left(\left[e^x \cos 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x (-2\sin 2x) \, dx \right.$$

$$= -2 \left(-e^{\frac{\pi}{2}} - 1 + 2\int_0^{\frac{\pi}{2}} e^x \sin 2x \, dx \right)$$

$$= 2e^{\frac{\pi}{2}} + 2 - 4\int_0^{\frac{\pi}{2}} e^x \sin 2x \, dx$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \sin 2x = \frac{2e^{\frac{\pi}{2}} + 2}{5}$$

8.4 Definite Integration of Odd and Even Functions

Definition:

Odd Function

If $f(x)$ is a function that satisfies $f(-x)=-f(x)$ for all values of x, $f(x)$ is an odd function.

Even Function

If $f(x)$ is a function that satisfies $f(-x)=f(x)$ for all values of x, $f(x)$ is an even function.

Example

18. Determine whether each of the following is an odd function or an even function.

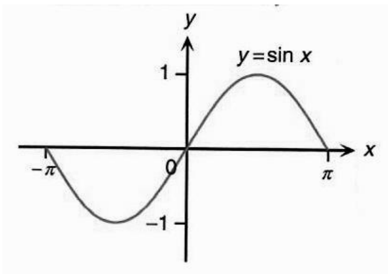
(a) $f(x)=\sin x$

odd function

$f(-x) = \sin(-x)$

$= -\sin x$

$= -f(x)$



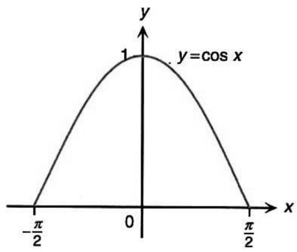
(b) $f(x)=\cos x$

even function.

$f(-x) = \cos(-x)$

$= \cos x$

$= f(x)$



Theorem

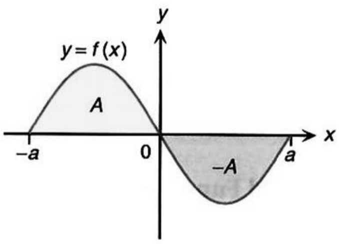
(a) If $f(x)$ is a continuous odd function, then

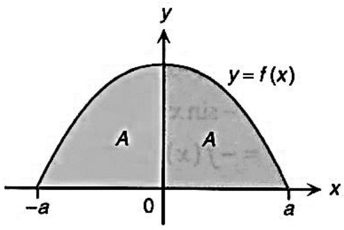
$\int_{-a}^{a} f(x)dx = 0$ for any constant a.

直接用

(b) If $f(x)$ is a continuous even function, then

$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ for any constant a.





Example

19. It is given that $\mathrm{f}(x)$ is a continuous even function. If $\int_{-7}^{7}\mathrm{f}(x)\mathrm{d}x = 11$ and $\int_{0}^{3}\mathrm{f}(x)\mathrm{d}x = 2$, evaluate the following definite integrals.

(a) $\int_{0}^{7}\mathrm{f}(x)\mathrm{d}x$

$= \frac{1}{2}\int_{-7}^{7} f(x)\,dx$

$= \frac{1}{2}\cdot 11$

$= \frac{11}{2}$

(b) $\int_{-3}^{7}\mathrm{f}(x)\mathrm{d}x$

$= \int_{-3}^{0} f(x)\,dx + \int_{0}^{7} f(x)\,dx$

$= \int_{0}^{3} f(x)\,dx + \frac{11}{2}$

$= 2 + \frac{11}{2}$

$= \frac{15}{2}$

20. (a) Let $\mathrm{f}(x) = x^5\cos x$. Show that $\mathrm{f}(x)$ is an odd function.

(b) Hence, evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x^5+1)\cos x\,\mathrm{d}x$.

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^5\cos x\,dx = 0$

(a) $f(-x) = (-x)^5\cos(-x)$

$= -x^5\cos x$

$= -f(x)$

$\therefore f(x)$ is odd

(b) $\cancel{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^5\cos x\,dx}^{\,0} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$

$= [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 1-(-1)$

$= 2$

21. (a) Let $f(x)$ be an odd function for $-p \le x \le p$, where p is a positive constant.

Prove that $\int_0^{2p} f(x-p)\,dx = 0$.

Hence evaluate $\int_0^{2p} [f(x-p)+q]\,dx$, where q is a constant.

(b) Prove that $\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)} = \dfrac{1+\sqrt{3}\tan x}{2}$.

$\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

(c) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,dx$.

(a) $\int_0^{2p} f(x-p)\,dx$

$= \int_{-p}^{p} f(u)\,du$

$= 0$

Let $u = x-p$

$du = dx$

when $x=0,\ u=-p$

$x = 2p,\ u = p$

$\int_0^{2p} [f(x-p)+q]\,dx$

$= \int_0^{2p} f(x-p)\,dx + \int_0^{2p} q\,dx$

$= 0 + [qx]_0^{2p}$

$= 2pq$

21. (a) Let $f(x)$ be an odd function for $-p \le x \le p$, where p is a positive constant.

Prove that $\int_0^{2p} f(x-p)\,dx = 0$.

Hence evaluate $\int_0^{2p} [f(x-p)+q]\,dx$, where q is a constant.

(b) Prove that $\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)} = \dfrac{1+\sqrt{3}\tan x}{2}$.

(c) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,dx$.

(b) L.H.S. $= \dfrac{\sqrt{3} + \dfrac{\tan x - \frac{1}{\sqrt{3}}}{1+\tan x \cdot \frac{1}{\sqrt{3}}}}{\sqrt{3} - \dfrac{\tan x - \frac{1}{\sqrt{3}}}{1+\tan x \frac{1}{\sqrt{3}}}}$

$= \dfrac{\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\tan x\right) + \tan x - \frac{1}{\sqrt{3}}}{\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\tan x\right) - \tan x + \frac{1}{\sqrt{3}})}$

$= \dfrac{\sqrt{3} + 2\tan x - \frac{1}{\sqrt{3}}}{\sqrt{3}+\frac{1}{\sqrt{3}}}$

$= \dfrac{\left(\dfrac{3+2\sqrt{3}\tan x - 1}{\sqrt{3}}\right)}{\left(\dfrac{3+1}{\sqrt{3}}\right)}$

$= \dfrac{2+2\sqrt{3}\tan x}{4}$

$= \dfrac{1+\sqrt{3}\tan x}{2}$

21. (a) Let $f(x)$ be an odd function for $-p \le x \le p$, where p is a positive constant.

Prove that $\int_0^{2p} f(x-p)\,dx = 0$.

Hence evaluate $\int_0^{2p} [f(x-p)+q]\,dx$, where q is a constant.

(b) Prove that $\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)} = \dfrac{1+\sqrt{3}\tan x}{2}$.

$2p = \frac{\pi}{3}$

$p = \frac{\pi}{6}$

(c) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,dx$.

$f(x-p)$

$= f\left(x-\frac{\pi}{6}\right)$

(c) $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,dx$

$= \int_0^{\frac{\pi}{3}} \ln 2 + \ln\left[\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)}\right] dx$

Let $f(x) = \ln\left(\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right)$

$f(-x) = \ln\left(\dfrac{\sqrt{3}+\tan(-x)}{\sqrt{3}-\tan(-x)}\right)$

$= \ln\left(\dfrac{\sqrt{3}-\tan x}{\sqrt{3}+\tan x}\right)$

$= \ln\left(\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right)^{-1}$

$= -\ln\left(\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right) = -f(x)$

$\therefore \int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,dx$

$= 2 \cdot \frac{\pi}{6} \cdot \ln 2$

$= \frac{\pi}{3}\ln 2$