Chapter 20 Arithmetic and Geometric Sequences
Supplementary Notes

Name: ________________________________ (   )   Class: ________

20.1 Revision on Sequences

A list of numbers arranged in order is called a sequence. Each number in the sequence is called a

term.

Examples of sequences:

(i) 2, 4, 6, 8, 10, 12   (ii) 3, −9, 27, −81, 243   (iii) 1, 7, 20, 12, 3

Note that the terms in a sequence may or may not form a pattern.

General Term of a Sequence

Consider the sequence

T_n 7, 13, 19, 25

Using $T(n)$ to denote the nth term in the sequence, we have

T_1 $T(1)=$ 7 , $T(2)=$ 13 , $T($3$)=$ 19 , $T($4$)=$ 25

In fact, the nth term of the sequence $T(n)$ is called the general **term** of the sequence.

From observation, the general term $T(n)=$ $6n+1$.

Hence, find the 10th term and 100th term of the sequence.

$T(10)=6\times10+1=61$, $T(100)=6\times100+1=601$

<u>Example</u>

1. The general term T_n of a sequence is $1-an$, where a is a constant. It is given that $T_4=-23$
   (a) Find the value of a.
   (b) Find T_8.
   (c) If $T_k=-65$, find k.

(a) $1-4a=-23$

$a=6$

(b) $T_n=1-6n$

$T_8=1-6\times8$

$=-47$

(c) $1-6k=-65$

$-6k=-66$

$k=11$

$n \to$ positive integer

2. The general term a_n of a sequence is $kn^2 - 1$, where k is a constant. It is given that $a_2 = 11$

   (a) Find a_{10}.

   (b) Determine whether 143 is a term of the sequence. Explain your answer.

(a) $k \cdot 2^2 - 1 = 11$

$4k = 12$

$k = 3$

(b) $3n^2 - 1 = 143$

$n^2 = 48$

$n = \pm\sqrt{48}$, not a positive integer.

$\therefore$ 143 is not a term in the sequence.

3. Let a_n be the nth term of a sequence. If $a_1 = 3$ and $a_{n+1} = 2a_n - 1$ for any positive integer n, find a_5.

$a_2 = 2a_1 - 1 = 2 \cdot 3 - 1 = 5$

$a_3 = 2a_2 - 1 = 2 \cdot 5 - 1 = 9$

$a_4 = 2a_3 - 1 = 2 \cdot 9 - 1 = 17$

$a_5 = 2a_4 - 1 = 2 \cdot 17 - 1 = 33$

4. Let a_n be the nth term of a sequence. If $a_1 = 2$, $a_2 = 5$ and $a_{n+2} = 2a_{n+1} + a_n$ for any positive integer n, find a_6.

$a_3 = 2a_2 + a_1 = 2 \cdot 5 + 2 = 12$

$a_4 = 2a_3 + a_2 = 2 \cdot 12 + 5 = 29$

$a_5 = 2a_4 + a_3 = 2 \cdot 29 + 12 = 70$

$a_6 = 2 \cdot a_5 + a_4 = 2 \cdot 70 + 29 = 169$

20.2 Arithmetic Sequences (AS)

A. Concept of Arithmetic Sequences

Definition: An arithmetic sequence is a sequence having a common **difference** between any term (except the first term) and its preceding term.

Example of an Arithmetic Sequence (or in short form A.S.):

$\boxed{+3}$ $\boxed{+3}$ $\boxed{+3}$ $\boxed{+3}$ $\boxed{+3}$

5, 8, 11, 14, 17, 20, …

後減前 = Constant

Common Difference = +3

Note that the common difference may be **positive, negative** or **zero**.

Example

5. Determine whether each of the following sequences is an arithmetic sequence. If it is, write down its common difference.

Sequence	Is it an Arithmetic Sequence?	Common Difference (if exist)
4, 9, 14, 19	Yes	5
2, −1, −4, −7	Yes	−3
12, 4, −4, −8 −8 −8 −4	No	/
3, $\frac{8}{3}$, $\frac{7}{3}$, 2	Yes	$-\frac{1}{3}$
2, 2, 2, 2	Yes	0

Class Activity

Consider an arithmetic sequence with the first term a and the common difference d.

$T_1 = a$

$T_2 = a + d$

$T_3 =$ $a+2d$

$T_4 =$ $a+3d$

…

$T_n =$ $a+(n-1)\cdot d$

The general term of an arithmetic sequence is given by

$$T_n = a + (n-1)d\ ,$$

where a is the first term and d is the common difference of the sequence.

Example

6. Given an arithmetic sequence 6, 15, 24, 33, …, find
   (a) the general term T_n,
   (b) T_{12}.

(a) $T_n = 6 + (n-1)\cdot 9$
$= 9n - 3$

(b) $T_{12} = 9 \times 12 - 3 = 105$

7. Consider the arithmetic sequence –11, –18, –25, –32, ….
   (a) Find the general term of the sequence,
   (b) Find the 10th term of the sequence.
   (c) If the mth term of the sequence is –109, find the value of m.

(a) $T_n = -11 + (n-1)(-7)$
$= -4 - 7n$

(b) $T_{10} = -4 - 7 \times 10 = -74$

(c) $-4 - 7m = -109$
$m = 15$

8. Consider the arithmetic sequence 27, … , –3, –9, … , –129.
   (a) Find the number of terms in the sequence. $n = ?$
   (b) Determine whether –42 is a term of the sequence. Explain your answer.

(a) $27 + (n-1)(-6) = -129$
$n - 1 = 26$
$n = 27$
$\therefore$ there are 27 terms

(b) $27 + (n-1)(-6) = -42$
$-6n + 6 = -69$
$n = 12.5$, not a positive integer
$\therefore$ -42 is not a term in the sequence

9. In the arithmetic sequence 172, 164, 156, …, $a = 172$, $d = -8$

(a) how many positive terms are there?

(b) Find the first negative term.

(a) $172 + (n-1)(-8) > 0$

$172 - 8n + 8 > 0$

$180 > 8n$

$22.5 > n$

$\therefore$ there are 22 positive terms

(b) first negative term

$= T_{23}$

$= 172 + (23-1)(-8)$

$= -4$

10. The 7th term and the 20th term of an arithmetic sequence are 4 and –22 respectively.

(a) Find the first term and the common difference of the sequence.

(b) Find the 13th term of the sequence.

(a) $a + 6d = 4$

$a + 19d = -22$

Solving, $a = 16$, $d = -2$

(b) $T_{13} = 16 + 12 \times (-2)$

$= -8$

11. In an arithmetic sequence, the 6th term is –57 and the sum of the first 2 terms is –186.

(a) Find the general term T_n.

(b) Which term is 39?

(a) $a + 5d = -57$ …(1)

$a + a + d = -186$

$2a + d = -186$ …(2)

$a = -97$, $d = 8$

$T_n = -97 + (n-1) \cdot 8$

$= 8n - 105$

(b) $8n - 105 = 39$

$n = 18$

$\therefore T_{18} = 39$

B. Properties of an Arithmetic Sequence

Property 1

If T_{n-1}, T_n, T_{n+1} are three consecutive terms of an arithmetic sequence, then

$T_{n+1} - T_n = T_n - T_{n-1}$

$$T_n = \frac{T_{n-1} + T_{n+1}}{2}.$$

$T_{n+1} + T_{n-1} = 2T_n$

Property 2

A sequence is an Arithmetic Sequence if the terms are selected at a regular interval from an Arithmetic Sequence.

Example: 2, 7, 12, 17, 22, 27, ...

2, 12, 22, ...

$kT_2 + c - (kT_1 + c)$

$= kT_2 - kT_1 = k(T_2 - T_1) = kd$

Property 3

If T_1, T_2, T_3, ... is an arithmetic sequence with common difference d, then $kT_1 + c$, $kT_2 + c$, $kT_3 + c$, ... is also an arithmetic sequence (where k and c are constants) with common difference $\underline{kd}$.

Example

12. If $2-3b$, $5-b$, $5+2b$ are three consecutive terms of an arithmetic sequence, find the value of b.

$5 - b - (2-3b) = 5 + 2b - (5-b)$ definition

$3 + 2b = 3b$

$b = 3$

13. Given that 31, a, $4a-83$, $4+2b$ is an arithmetic sequence, find the values of a and b.

$a - 31 = 4a - 83 - a$

$a = 26$

$4 + 2b - (4a - 83) = 4a - 83 - a$

$2b - 17 = -5$

$b = 6$

14. It is given that $\log x$, $\log 6$ and $\log(x+5)$ is an arithmetic sequence. Find x.

$$\log 6 - \log x = \log(x+5) - \log 6$$
$$\log \frac{6}{x} = \log \frac{x+5}{6}$$
$$\frac{6}{x} = \frac{x+5}{6}$$

$$36 = x^2 + 5x$$
$$0 = x^2 + 5x - 36$$
$$x = 4 \text{ or } -9 \text{ (rej.)}$$

M.C.

$c - b = b - a = d$

15. Suppose a, b, c is an arithmetic sequence with common difference d. Complete the following table below in terms of d.

Sequence	Arithmetic Sequence?	Common Difference (if exist)
$a+10$, $b+10$, $c+10$	Yes	d
$-2a$, $-2b$, $-2c$	Yes	$-2d$
$\sqrt{b}-\sqrt{a} = \sqrt{c}-\sqrt{b}$ $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$	No	/
$5a+1$, $5b+1$, $5c+1$	Yes	$5d$
$b-c = a-b = -d$ c, b, a	Yes	$-d$

1.3 Geometric Sequence

A. Concept of Geometric Sequences

Definition:

A geometric sequence is a sequence having a common ratio between any term (except the first term) and its preceding term.

Example of a Geometric Sequence (or in short form G.S.):

×5 ×5

2, 10, 50, 250, 1250, 6250, …

×5 ×5 ×5

Common Ratio = $+5$

Note that the common ratio may be positive or negative, but not zero.

Example

16. Determine whether each of the following sequences is a geometric sequence. If it is, write down its common ratio.

Sequence	Is it a Geometric Sequence?	Common Ratio (if exist)
8, 16, 32, 64	Yes	2
$\frac{1}{2}$, $-\frac{1}{4}$, $\frac{1}{8}$, $-\frac{1}{16}$	Yes	$-\frac{1}{2}$
A.S + G.S. → 5, 5, 5, 5, 5	Yes	1
$\frac{2}{3}$, $\frac{2}{5}$, $\frac{6}{25}$, $\frac{18}{125}$	Yes	$\frac{3}{5}$
2, –4, 8, 16	No	/

$\times -2$ $\quad \times -2$ $\quad \times +2$

Class Activity

Consider a geometric sequence with the first term a and the common ratio r.

$T_1 = a$

$T_2 = ar$

$T_3 =$ $ar \cdot r = ar^2$

$T_4 =$ $ar^2 \cdot r = ar^3$

…

$T_n =$ ar^{n-1}

The general term of a geometric sequence is given by

$$T_n = ar^{n-1},$$

where a is the first term and r is the common ratio of the sequence.

次方 ⇒ log.

Example

16. Given a geometric sequence 4, 12, 36, 108, ….
    (a) Find the general term T_n,
    (b) If the kth term of the sequence is 2916, find k.

(a). $a = 4$, $r = 3$

$T_n = 4 \cdot 3^{n-1}$

(b) $4 \cdot 3^{k-1} = 2916$

$3^{k-1} = 729$

$(k-1) \log 3 = \log 729$

$k - 1 = 6$

$k = 7$

17. How many terms are there in the geometric sequence $\frac{3}{16}$, $-\frac{3}{4}$, 3, … , −192?

$$\frac{3}{16}\cdot(-4)^{n-1} = -192$$

$$(-4)^{n-1} = -1024 \rightarrow n-1 \text{ is odd}$$

$$(n-1)\log(-4) = \log(-1024) \quad X$$

$$4^{n-1} = 1024 \quad \checkmark$$

$$(n-1)\log 4 = \log 1024$$

$$n-1 = 5$$

$$n = 6$$

18. Suppose the geometric sequence 162, …, $\frac{2}{27}$ has 8 terms.

(a) Find the common ratio of the sequence.

(b) Find the 5th term of the sequence.

(a) $162\cdot r^7 = \frac{2}{27}$

$r^7 = \frac{1}{2187}$

$r = \frac{1}{3}$

(b) $T_5 = 162\cdot\left(\frac{1}{3}\right)^4$

$= 2$

2 conditions → 2 unknowns → 2 equations

* 19. The 2nd term and the 5th term of a geometric sequence are 6 and ~~108~~ 162 respectively.

(a) Find the first term of the sequence.

(b) Find the 9th term of the sequence.

(a) $ar = 6 \quad \cdots (1)$

$ar^4 = 162 \quad \cdots (2)$

$\frac{(2)}{(1)}: \quad \frac{ar^4}{ar} = 27$

$r^3 = 27$

$r = 3$

$a = 6 \div 3 = 2$

(b) T_9

$= 2\cdot 3^8$

$= 13122$

20. The 3rd term and the 7th term of a geometric sequence are 36 and 2916 respectively.
   (a) Find the first term and the common ratio of the sequence.
   (b) Find the 6th term of the sequence.

(a) $ar^2 = 36 \cdots (1)$

$ar^6 = 2916 \cdots (2)$

$\frac{(2)}{(1)}: \quad r^4 = 81$

$r = \pm 3$

$a = 36 \div r^2 = 4$

(b) when $r = 3$,

$T_6 = 4 \cdot 3^5 = 972$

when $r = -3$

$T_6 = 4 \cdot (-3)^5 = -972$

21. The sum of the first term and the 2nd term of a geometric sequence is –2, while the sum of the 4th term and the 5th term of the sequence is 54. Find the 8th term of the sequence.

$a + ar = -2$

$a(1+r) = -2 \quad \cdots (1)$

$ar^3 + ar^4 = 54$

$ar^3(1+r) = 54 \cdots (2)$

$\frac{(2)}{(1)}: \quad r^3 = -27$

$r = -3$

$a = 1$

$T_8 = 1 \cdot (-3)^7 = -2187$

22. Let a_n be the nth term of a geometric sequence. It is given that $a_1 = 2$ and $a_2 = 8$. Find the greatest term in the sequence which is smaller than 10^5.

$a = 2$

$r = 4$

$2 \cdot 4^{n-1} < 10^5$

$4^{n-1} < 50000$

$(n-1) \log 4 < \log 50000$

$n < 8.8048$

$\therefore$ required term

$= T_8$

$= 2 \cdot 4^7$

$= 32768$

22. Let a_n be the nth term of a geometric sequence. It is given that $5a_2 = 6a_3$ and $a_1 = 36$. Find the ~~greatest~~ smallest term in the sequence which is greater than $\frac{1}{100}$.

$\frac{5}{6} = \frac{a_3}{a_2}$

$r = \frac{5}{6}$

$36 \cdot \left(\frac{5}{6}\right)^{n-1} > \frac{1}{100}$

$\left(\frac{5}{6}\right)^{n-1} > \frac{1}{3600}$

$(n-1) \log \frac{5}{6} > \log \frac{1}{3600}$ $\ominus$

$n-1 < \log \frac{1}{3600} \div \log \frac{5}{6}$.

$n < 45.9134$

$\therefore$ required term

$= T_{45}$

$= 36 \cdot \left(\frac{5}{6}\right)^{44}$

$= 0.0118$

23. It is given that 10, 50, 250, … is a geometric sequence.
   (a) Find the general term T_n of the sequence.
   (b) Find the greatest value of n such that the sum of the nth term and the $2n$th term is less than 80400.

reduce to quadratic.

$5^n, 5^{2n} = (5^n)^2$

$\Rightarrow u, u^2$

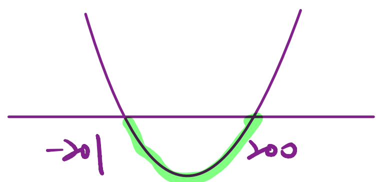
(a) $T_n = 10 \cdot 5^{n-1}$

(b) $10 \cdot 5^{n-1} + 10 \cdot 5^{2n-1} < 80400$

$10 \cdot \frac{5^n}{5} + 10 \cdot \frac{5^{2n}}{5} < 80400$

$5^{2n} + 5^n - 40200 < 0$

Let $u = 5^n$, $u^2 + u - 40200 < 0$



$-201 < u < 200$

$-201 < 5^n < 200$

$n \log 5 < \log 200$

$n < 3.2920$

$\therefore$ greatest $n = 3$

B. Properties of a Geometric Sequence

Property 1

If $T(n-1)$, $T(n)$, $T(n+1)$ are three consecutive terms of a geometric sequence, then

$$[T(n)]^2 = T(n-1)\times T(n+1).$$

$$\frac{T(n)}{T(n-1)} = \frac{T(n+1)}{T(n)}$$

Property 2

A sequence is a Geometric Sequence if the terms are selected at a regular interval from a Geometric Sequence.

Example: 2, 10, 50, 250, 1250, 6250

Property 3

If $T(1)$, $T(2)$, $T(3)$,... is a geometric sequence with common ratio r, then $kT(1)$, $kT(2)$, $kT(3)$,... is also a geometric sequence (where k is a constant) with common ratio r.

Example

24. If $8k$, $k + 1$, $2k$ are in geometric sequence, find the possible value(s) of k.

$$\frac{k+1}{8k} = \frac{2k}{k+1}$$

$$k^2+2k+1 = 16k^2$$

$$15k^2-2k-1 = 0$$

$$k = \frac{1}{3} \text{ or } -\frac{1}{5}$$

25. If 4, $10 - r$, $-10r$ is a geometric sequence, find the common ratio.

$$\frac{10-r}{4} = \frac{-10r}{10-r}$$

$$100 - 20r + r^2 = -40r$$

$$r^2 + 20r + 100 = 0$$

$$r = -10$$

M.C.

26. Suppose a, b, c is a geometric sequence with common ratio r. Complete the following table below in terms of r.

$\frac{b}{a} = \frac{c}{b} = r$

Sequence	Geometric Sequence?	Common ratio (if exist)
$-8a, -8b, -8c$	Yes	r
$a-5, b-5, c-5$	No	/
$\sqrt{a}, \sqrt{b}, \sqrt{c}$	Yes	$\sqrt{r}$
a^3, b^3, c^3	Yes	r^3
$\frac{b}{c} = \frac{a}{b} =$ c, b, a	Yes	$\frac{1}{r}$
$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$	Yes	$\frac{1}{r}$
$\frac{5^b}{5^a} = 5^{b-a}$ $5^a, 5^b, 5^c$ $\frac{5^c}{5^b} = 5^{c-b}$	No	/

27. It is given that a, b, c are three consecutive terms of a geometric sequence. Show that $\log a^3$, $\log b^3$, $\log c^3$ are three consecutive terms of an arithmetic sequence.

$\frac{b}{a} = \frac{c}{b} = r$

$\log b^3 - \log a^3 = 3\log b - \log a$
$= 3\left(\log \frac{b}{a}\right)$
$= 3\log r$

$\log c^3 - \log b^3 = 3\left(\log \frac{c}{b}\right) = 3\log r$

28. It is given that that a, b, c are three consecutive terms of an arithmetic sequence. Show that 2^a, 2^b, 2^c are three consecutive terms of a geometric sequence.

$b - a = c - b = d$

$2^b \div 2^a = 2^{b-a} = 2^d$

$2^c \div 2^b = 2^{c-b} = 2^d$

29. Suppose p, 9, q are the first 3 terms of an arithmetic sequence and p, 6, q are the first 3 terms of a geometric sequence, find the value of p^2+q^2.

$q-9=9-p$

$p+q=18$

$\frac{6}{p}=\frac{q}{6}$

$pq=36$

p^2+q^2

$=(p+q)^2-2pq$

$=18^2-72$

$=252$

30. Suppose h, 4, k form a geometric sequence and k, h, 4 form an arithmetic sequence, where h and k are non-zero real numbers. Find the values of h and k.

$\frac{4}{h}=\frac{k}{4}$

$16=hk$

$h-k=4-h$

$2h=4+k$

$16=h(2h-4)$

$16=2h^2-4h$

$h^2-2h-8=0$

$h=4$ or -2

when $h=4$, $k=4$

$h=-2$, $k=-8$

Summary

1. Arithmetic Sequences

- An arithmetic sequence is a sequence in which there is a common difference d between two adjacent terms. ($T_n-T_{n-1}=d$ for $n=2, 3, 4, \ldots$)
- For an arithmetic sequence with first term a and common difference d, its general term is:

$$T_n=a+(n-1)d\text{, where } n \text{ is a positive integer.}$$

2. Geometric Sequences

- A geometric sequence is a sequence in which there is a common ratio r between two adjacent terms. ($\frac{T_n}{T_{n-1}}=r$ for $n=2, 3, 4, \ldots$)
- For a geometric sequence with first term a and common ratio r, its general term is:

$$T_n=ar^{n-1}\text{, where } n \text{ is a positive integer and } r\neq 0.$$

20.4 Applications of Arithmetic and Geometric Sequences

Example

A.S. → $d=4$

31. In a supermarket, cans of soft drink are stacked in a total of 15 layers. Each layer, except the top layer, has 4 more cans than the layer above it. It is given that there are 38 cans of soft drink in the 9th layer counting from the top.
    (a) Find the number of cans of soft drink in the bottom layer.
    (b) Which layer, counting from the top, has 30 cans of soft drink?

(a) $a+(9-1)\cdot 4 = 38$

$a = 6$

required no. of cans

$= 6+(15-1)\cdot 4$

$= 62$

(b) $6+(n-1)\cdot 4 = 30$

$n-1 = 6$

$n = 7$

32. Yannie spent $200 000 to buy a car at the beginning of 2011. The value of the car depreciated of 2011. The value of the car depreciated at a constant rate of 12% per year. Let $ P_n be the value of the car n years after the beginning of 2011.

G.S. → $r = (1-12\%) = 0.8$

    (a) Express P_n in terms of n.
    (b) At the beginning of which year will the value of the car first fall below $40 000?

(a) $P_n = \underbrace{200000 \times 0.88}_{a} \times \underbrace{0.88^{n-1}}_{r^{n-1}} = 200000 \times 0.88^n$

(b) $200000 \times 0.88^n < 40000$

$0.88^n < 0.2$

$n \log 0.88 < \log 0.2$

$n > 12.5901$

∴ at the beginning of 13th year

∴ at the beginning of 2024

33. In an organization, the amount spent on project A in the 1st year since the start of the project is $\$5\times10^6$ and in the subsequent years, the amount spent is r% more than that in the previous year. It is given that the amount spent on project A in the 3rd year is $\$7.2\times10^6$. $(1+44\%) = 1.44 = 1.2^2$

(a) Find r. 20 $(1+20\%) = 1.2$

(b) The amount spent on project B in the same organization in the 1st year since the start of the project is $\$3\times10^6$ and in the subsequent years, the amount spent is 44% more than that in the previous year. It is given that the two projects A and B start in the same year. In which year will the total amount spent on projects A and B first exceeds $\$2.8\times10^7$?

quadratic inequality.

(a) $5\times10^6\times(1+r\%)^2 = 7.2\times10^6$

$(1+r\%)^2 = 1.44$

$r = 20$

(b) $3\times10^6\times1.44^{n-1} + 5\times10^6\times1.2^{n-1} > 2.8\times10^7$

$\dfrac{3\times(1.2^n)^2}{1.44} + \dfrac{5\times1.2^n}{1.2} > 28$

$3\times(1.2^n)^2 + 6\times1.2^n - 40.32 > 0$

$1.2^n < -4.8$ (rej.) or $1.2^n > 2.8$

$n > 5.6473$

$\therefore$ in the 6th year.