Chapter 19 Linear Programming
Supplementary Notes

Name: ________________________________ (   ) Class: ________

19.1 Linear Inequalities in Two Unknowns

If the linear inequality contains two unknowns, it is called a **linear inequality in two unknowns.**
For example, $x > y + 2$, $y \le x$ are linear inequalities in two unknowns.

An ordered pair (x, y) that satisfies an inequality in two unknowns is a solution of the inequality.

Consider the following ordered pairs:

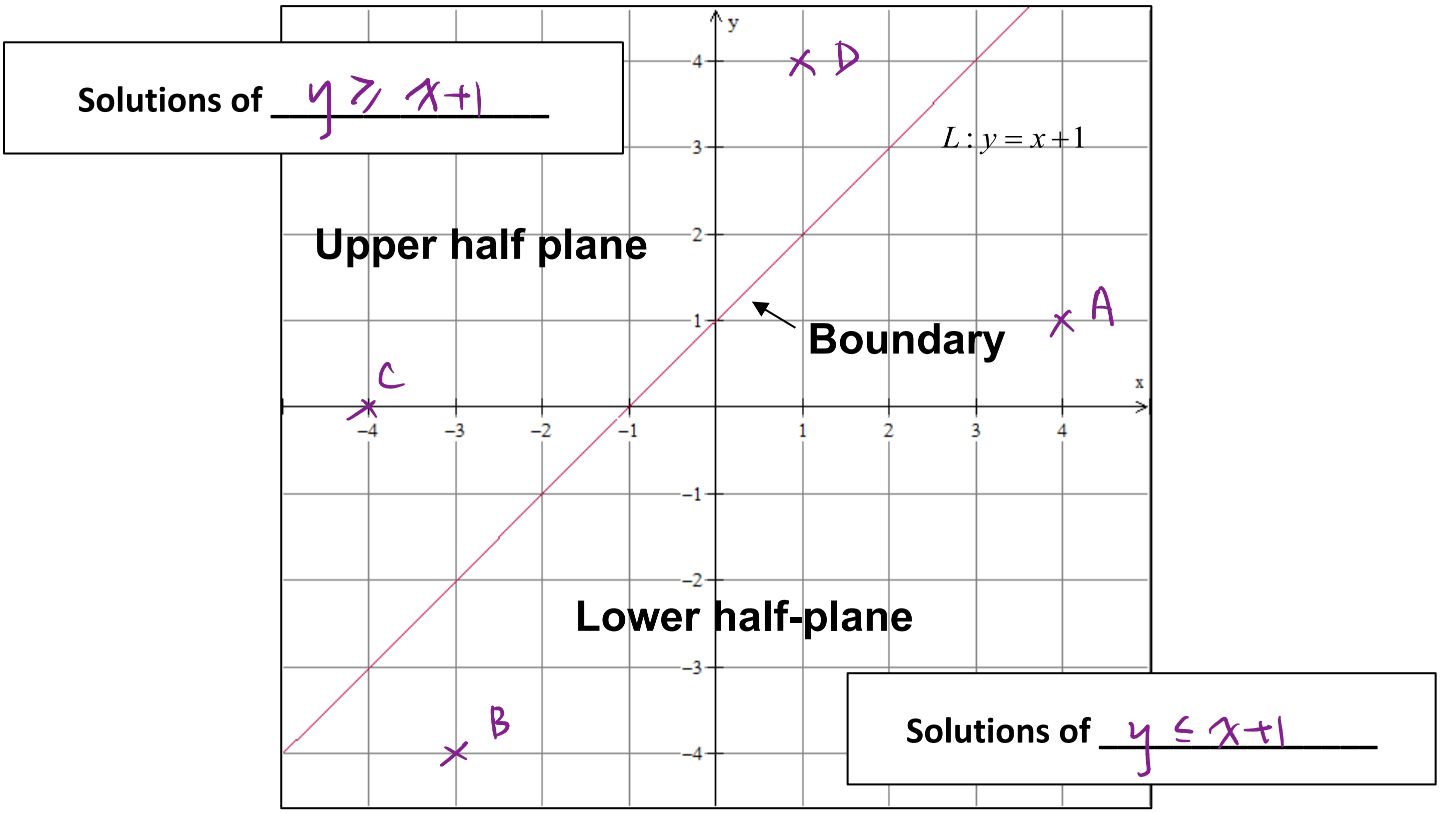
$A(4, 1)$, $B(-3, -4)$, $C(-4, 0)$, $D(1, 4)$

Solutions of $y \ge x+1$: C(-4, 0), D(1, 4)

Solutions of $y \le x+1$: A(4, 1), B(-3, -4)

In fact, each inequality has an infinite number of solutions. And all the solutions of a linear inequality in two unknowns can be represented graphically.

Consider the graph of $y = x+1$. It divides the coordinate plane into two regions.



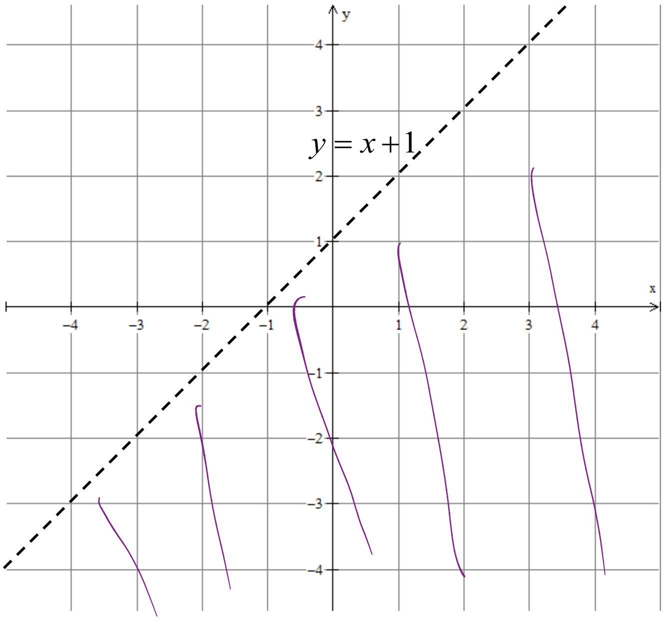
(0, 0)

Since the solutions of a linear inequality in two unknowns can be represented by one of the half-planes, we can choose a test point (**which does not lie on the boundary**) to check whether the half-plane containing the test point represents the solutions of the inequality.

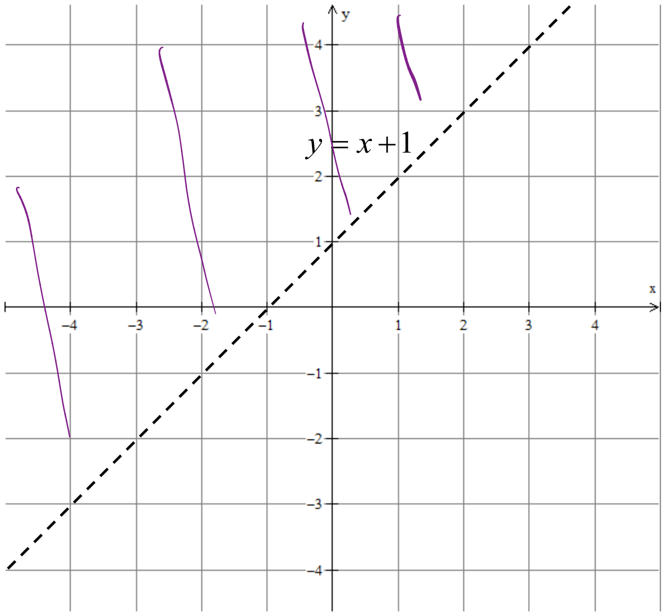
$>, <$ → dotted line

$\geq, \leq$ → solid line

Solutions of $y < x + 1$

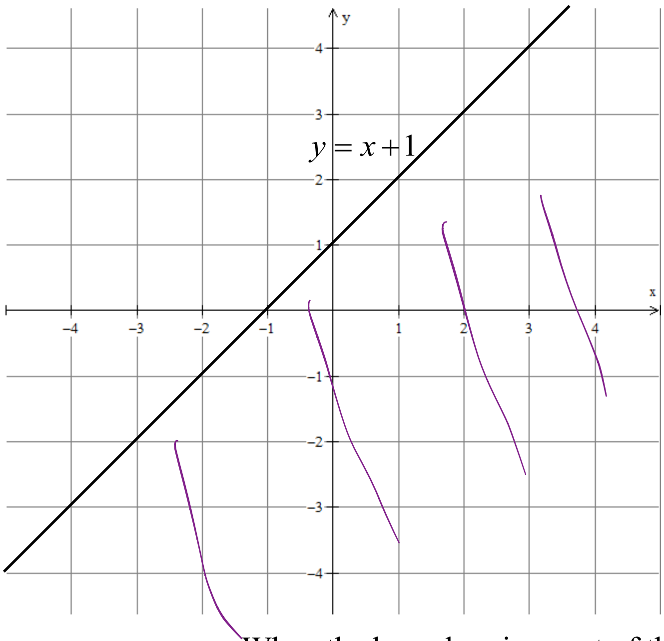


Solutions of $y > x + 1$

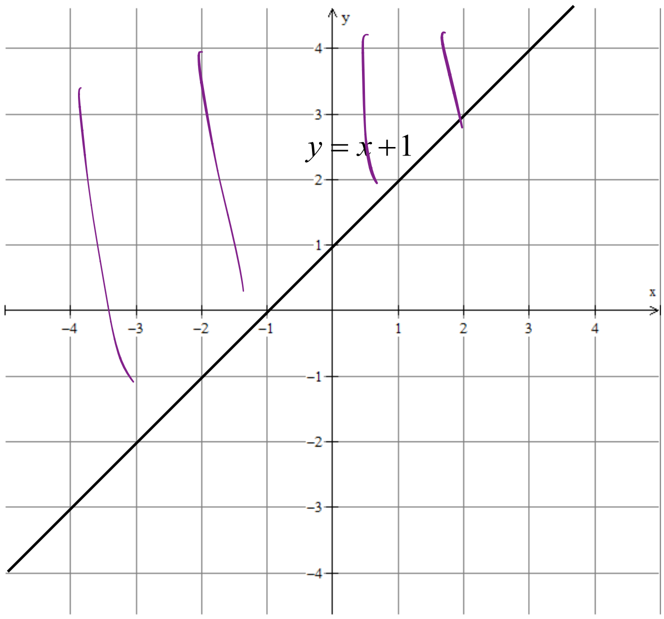


When the boundary is not a part of the solutions, it is drawn as **dotted line**.

Solutions of $y \leq x + 1$



Solutions of $y \geq x + 1$



When the boundary is a part of the solutions, it is drawn as **solid line.**

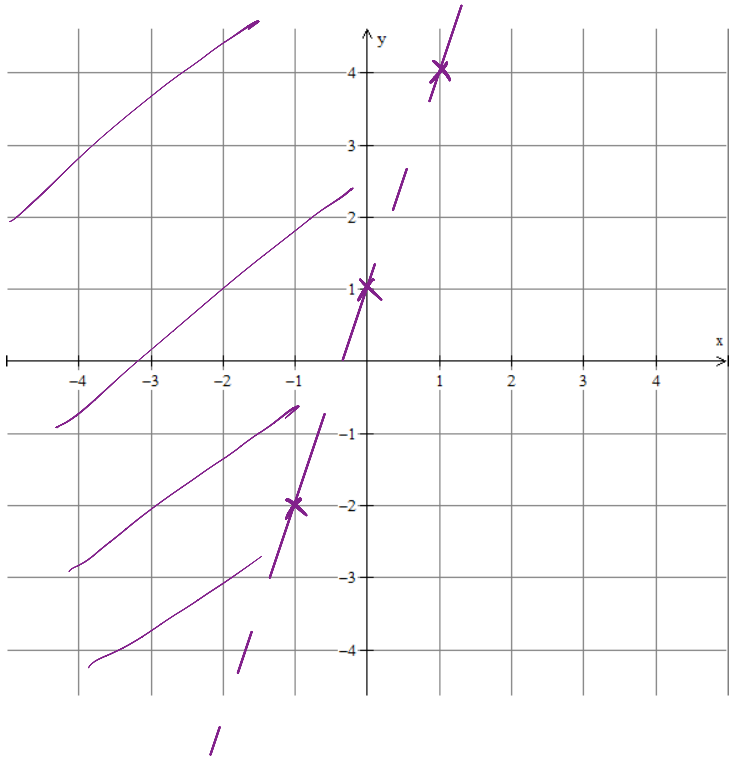
Steps	**Example:** Solve the inequality $y-3x>1$ graphically.

1. Draw the straight line $y-3x=1$
   $>$, $<$: Draw the **dotted line.**
   $\geq$, $\leq$: Draw the **solid line.**

$y-3x=1$

x	-1	0	1
y	-2	1	4

2. Choose a test point (0, 0) and check which half-plane represents the solutions of the inequality.
   * In fact, we can choose **any point** that does not lie on the boundary as the test point.



Test point (0, 0):

$y-3x$

$=$ 0 $-3($ 0 $)$

$=$ 0

<1

Shade the half-plane that contains / does not contain the test point (0, 0).

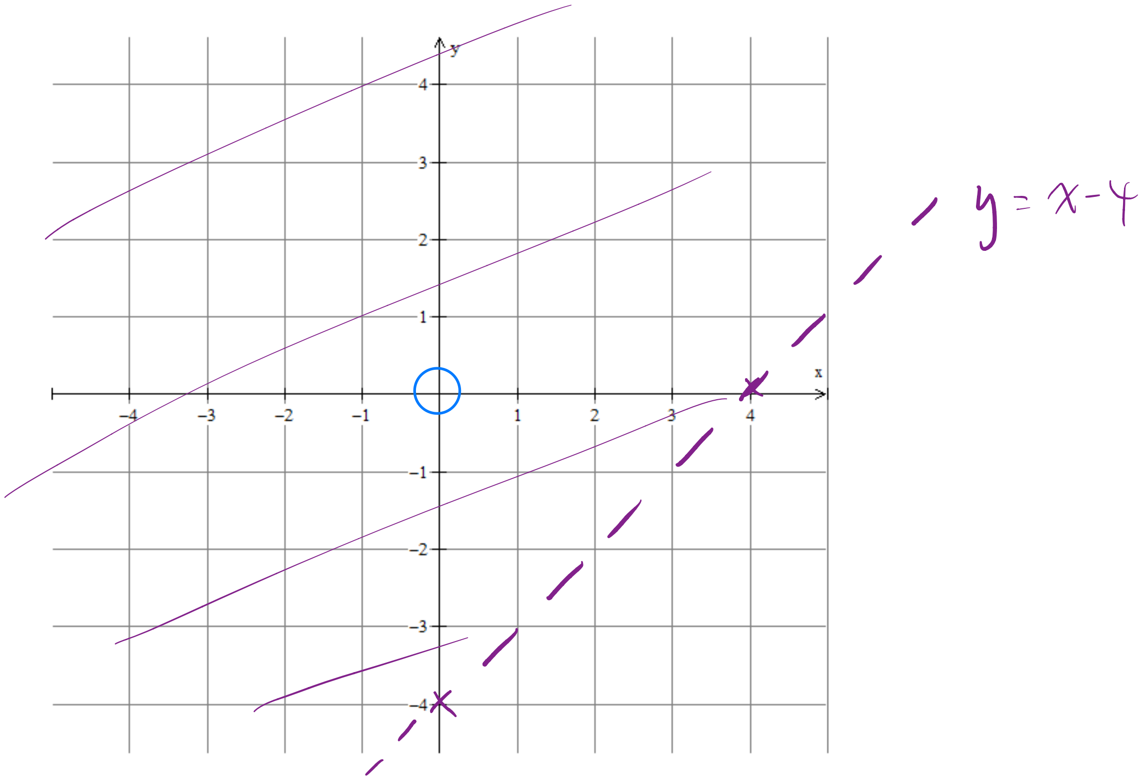
Example

1. Solve the following inequalities graphically.

(a) $y > x - 4$

$y = x - 4$

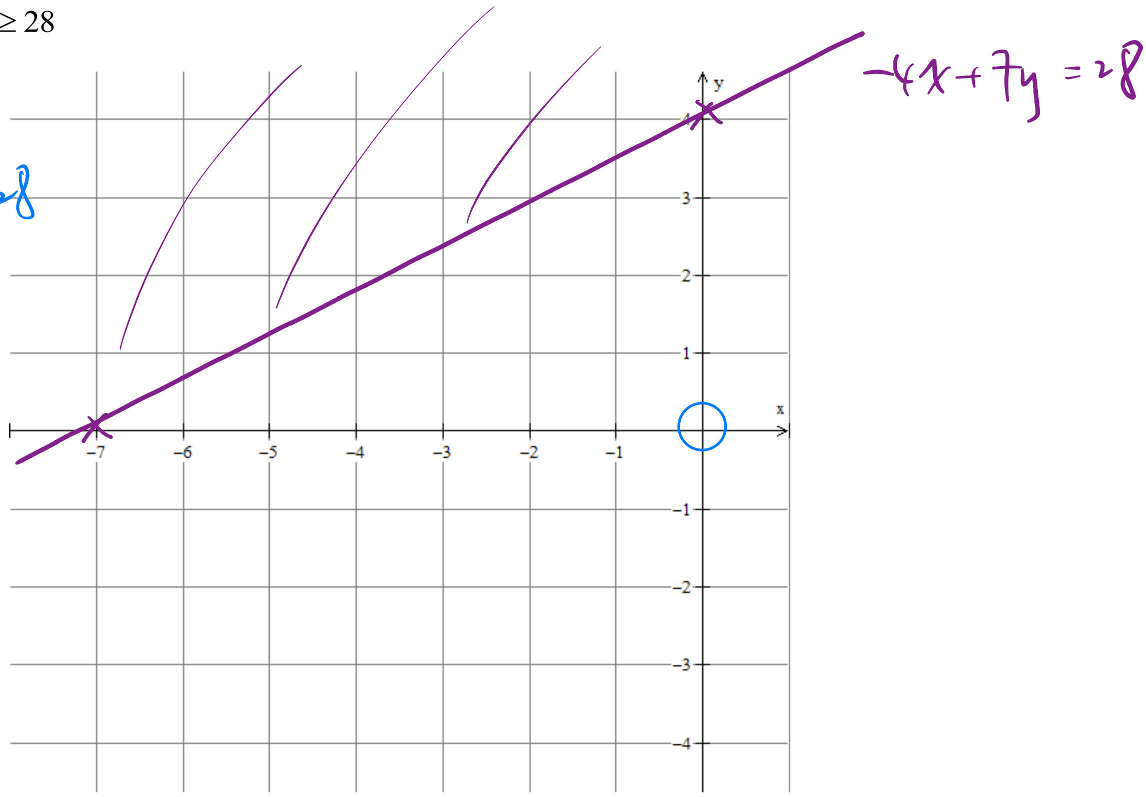
$0 > 0 - 4$



(b) $-4x + 7y \geq 28$

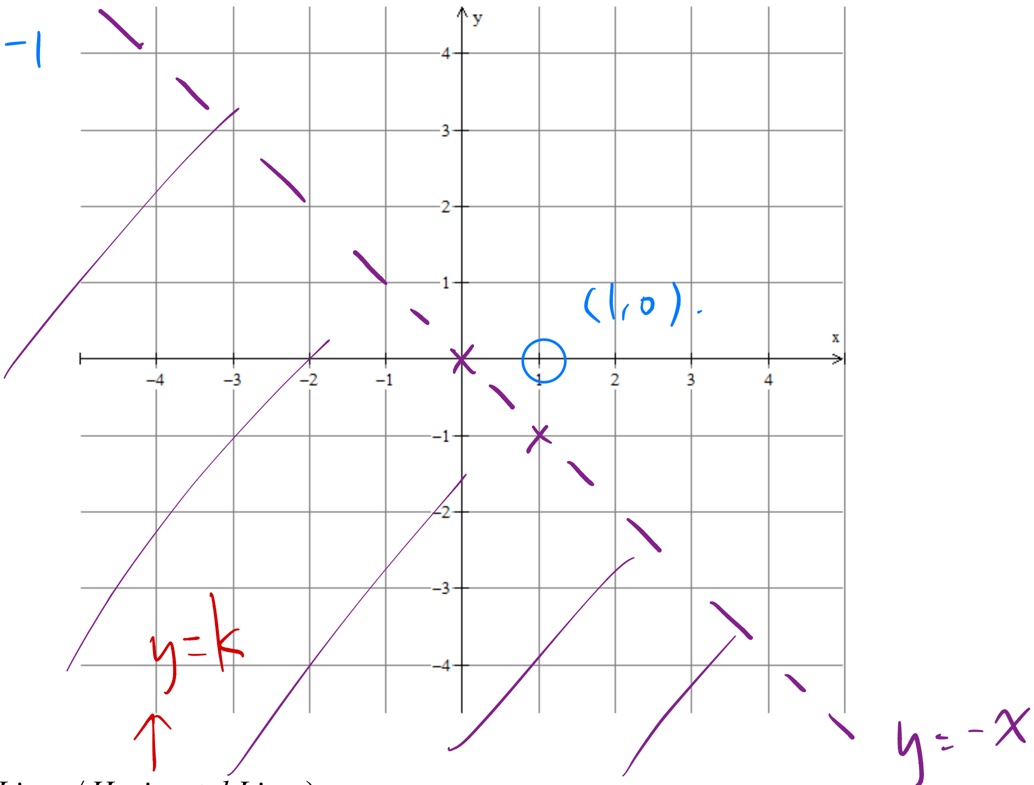
$-4x + 7y = 28$

$-4(0) + 7(0) < 28$



(c) $y < -x$

$0 > -1$

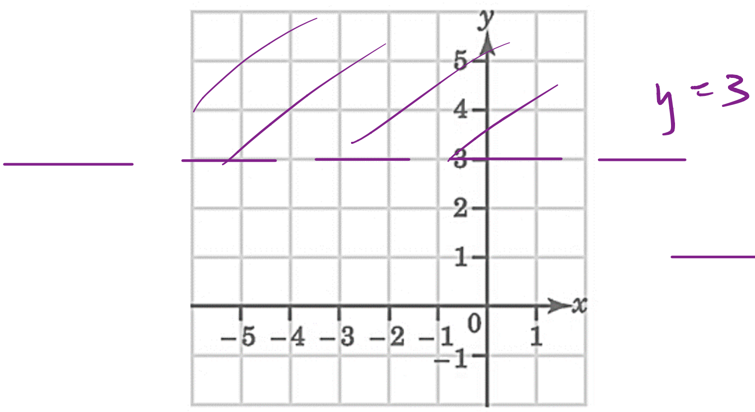


$x = k$

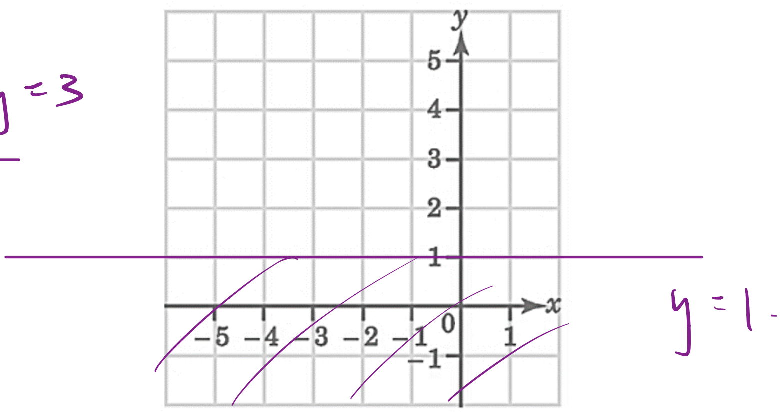
$y = k$

(*Vertical Lines* / *Horizontal Lines*)

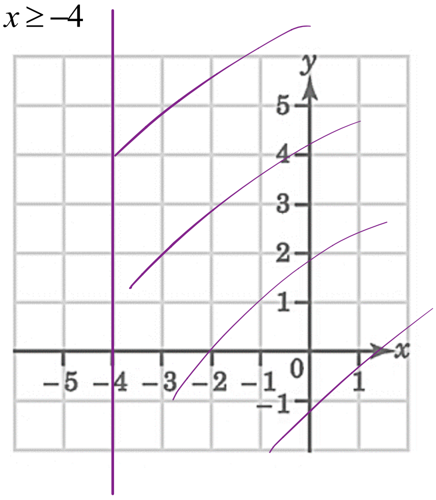
(d) $y > 3$



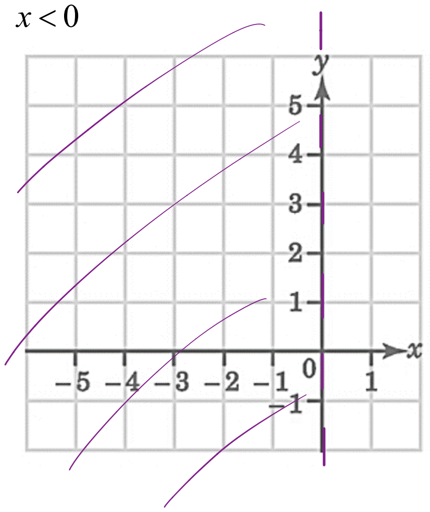
(e) $y \leq 1$



(f) $x \geq -4$

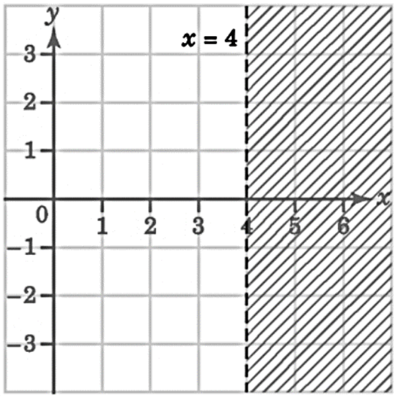


(g) $x < 0$



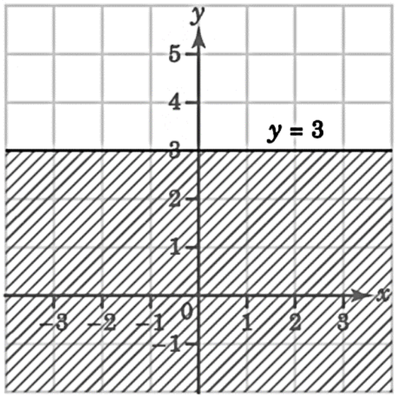
2. In each of the following, the shaded region represents the solutions of an inequality. Write down the inequality.

(a)



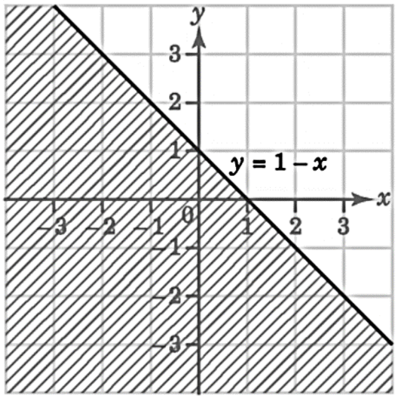
The required inequality: $x > 4$

(b)



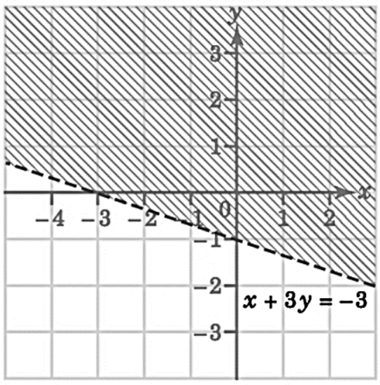
The required inequality: $y \leq 3$

(c)



The required inequality: $y \leq 1 - x$

(d)

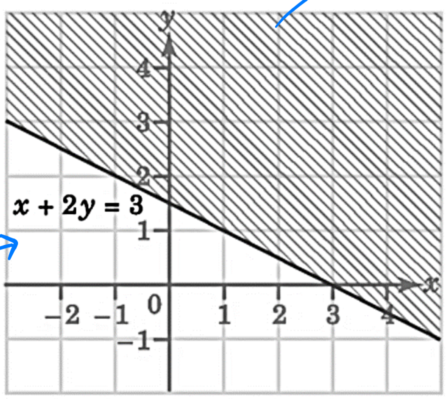


The required inequality: $x + 3y > -3$

(e)

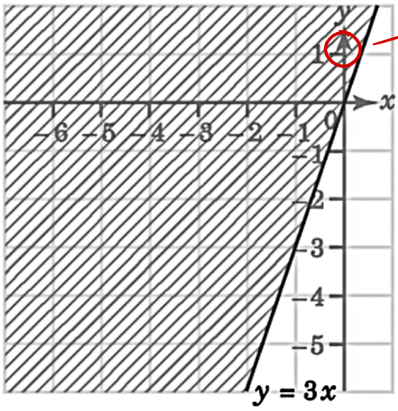
$x+2y \geq 3$

$x+2y \leq 3$



The required inequality: $x+2y \geq 3$

(f)



The required inequality: $y \geq 3x$

19.2 Systems of Linear Inequalities in Two Unknowns

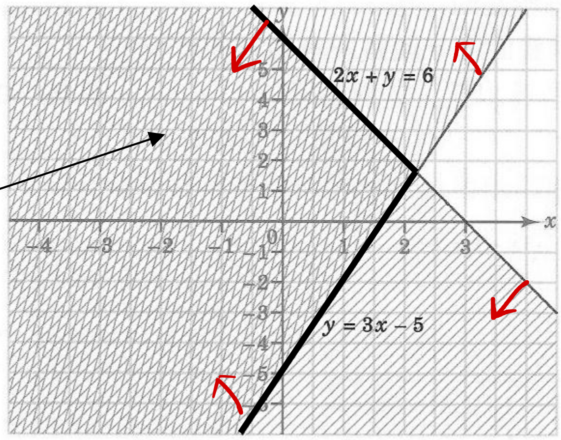
To solve a system of linear inequalities in x and y, we have to find all ordered pairs (x, y) satisfying **all the linear inequalities.**

Consider the following system of linear inequalities in two unknowns:

$$\begin{cases} y \ge 3x-5 & \leftarrow (0,0)\ \checkmark \\ 2x+y \le 6 & \leftarrow (0,0)\ \checkmark \end{cases}$$

Overlapping Region

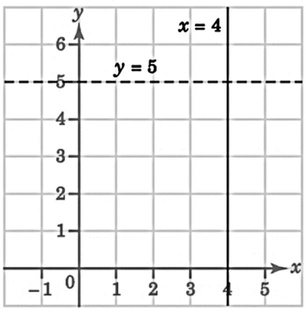
Feasible



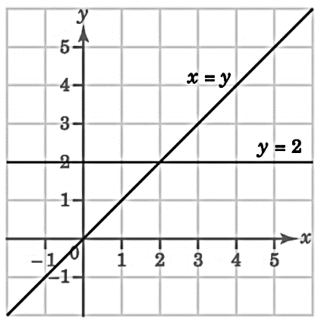
Example

3. In each of the following figures, shade the solution region of the given system of linear inequalities in two unknowns.

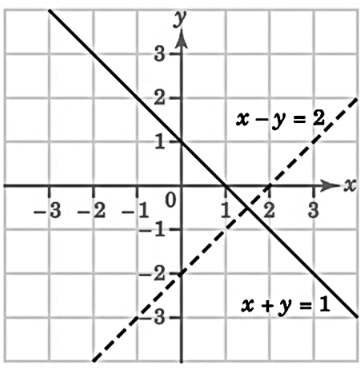
(a) $\begin{cases} x \le 4 \\ y < 5 \end{cases}$



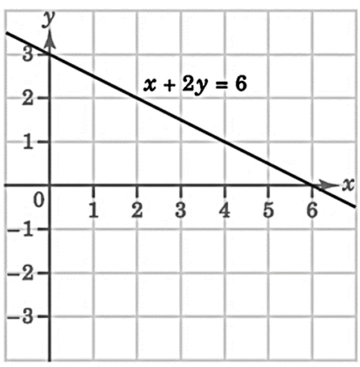
(b) $\begin{cases} y \ge 2 \\ x \ge y \end{cases}$



(c) $\begin{cases} x + y \ge 1 \\ x - y < 2 \end{cases}$



(d) $\begin{cases} x \ge 0 \\ y \ge 0 \\ x + 2y \le 6 \end{cases}$



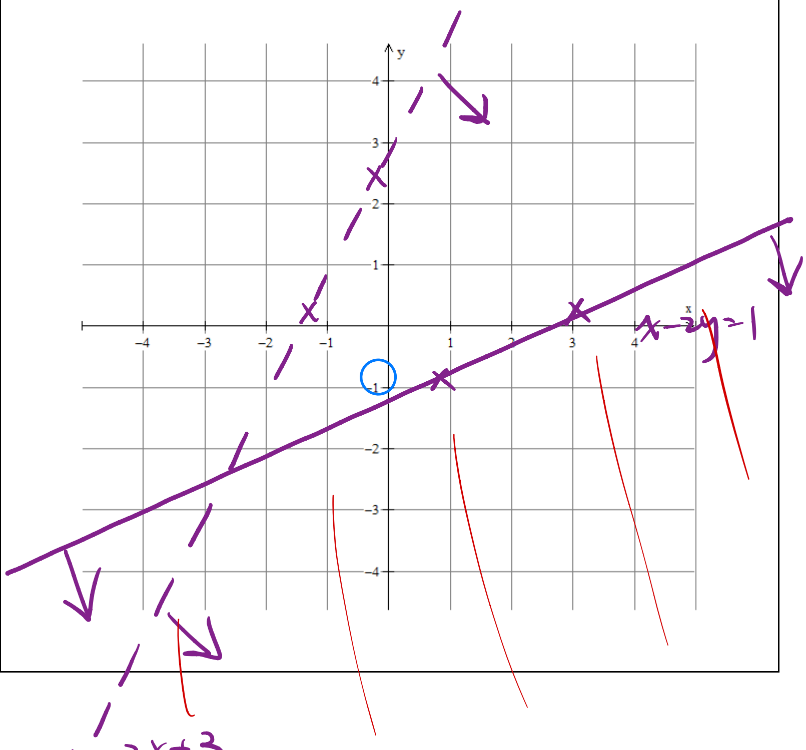
Steps	Example: Solve $\begin{cases} y < 2x+3 \\ x-2y \geq 1 \end{cases}$ graphically.
1. Draw the two straight lines correspond to the two inequalities: $y=2x+3$, $x-2y=1$	see tables below
2. Add **arrows** on the boundaries to indicate which half-plane represents the solutions of each individual inequality.	
3. Shade the overlapping region of the two graphical representations. It represents the solutions of the system of linear inequalities.	see graph below

$y=2x+3$

x	-1	0
y	1	3

$x-2y=1$

x	1	3
y	0	1



Example

4. In each of the following, shade the region that represents the solutions of the system of inequalities.

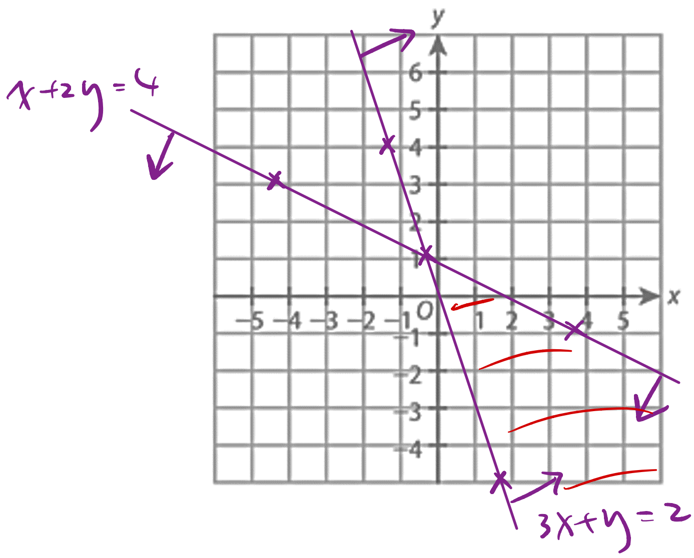
(a) $\begin{cases} x+2y \leq 4 \\ 3x+y \geq 2 \end{cases}$

$x+2y=4$

x	-4	0	4
y	4	2	0

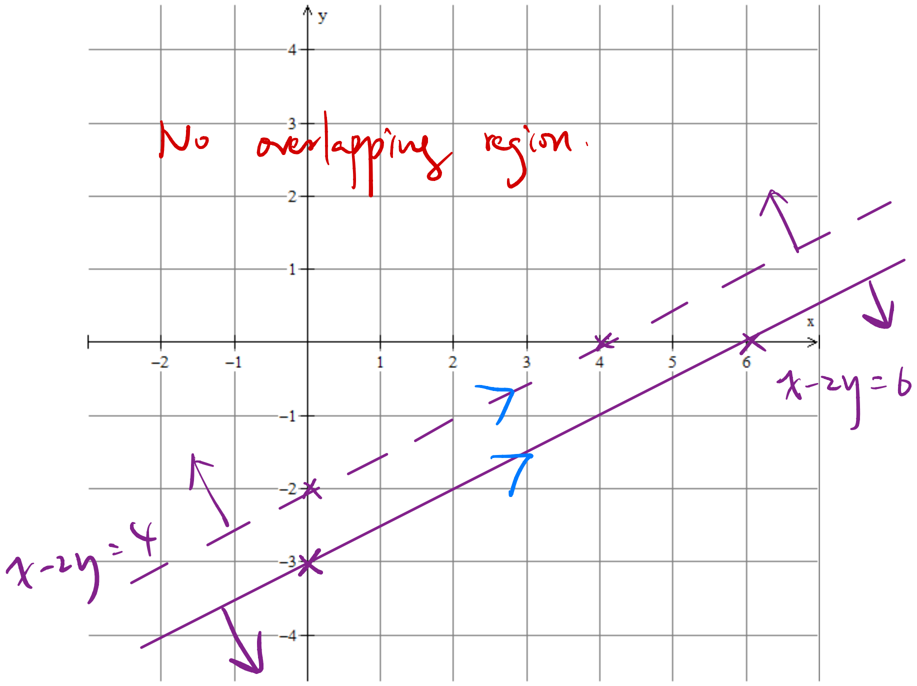
$3x+y=2$

x	-1	0	2
y	5	2	-4

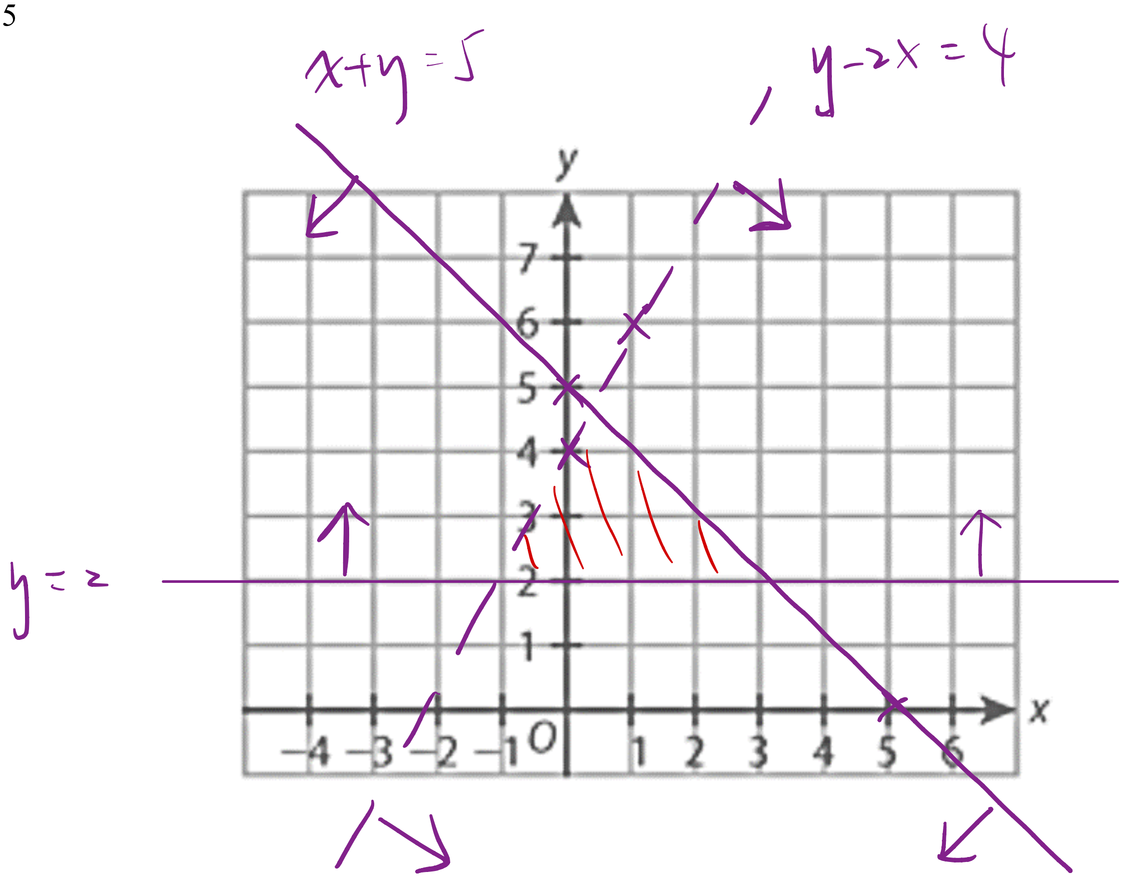


(b) $\begin{cases} x-2y \geq 6 \\ x-2y < 4 \end{cases}$

no solution.

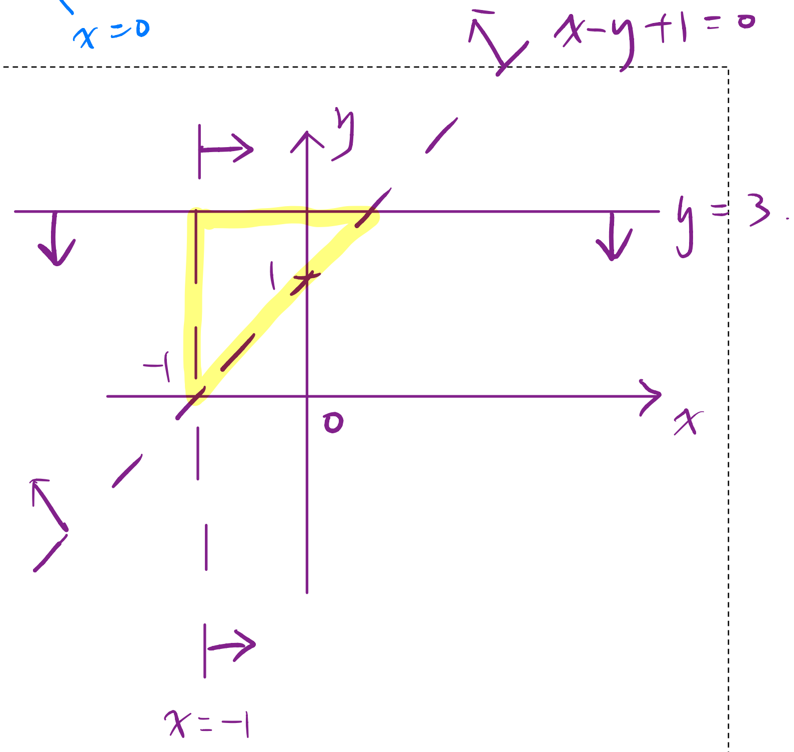


(c) $\begin{cases} y-2x < 4 \\ x+y \leq 5 \\ y \geq 2 \end{cases}$



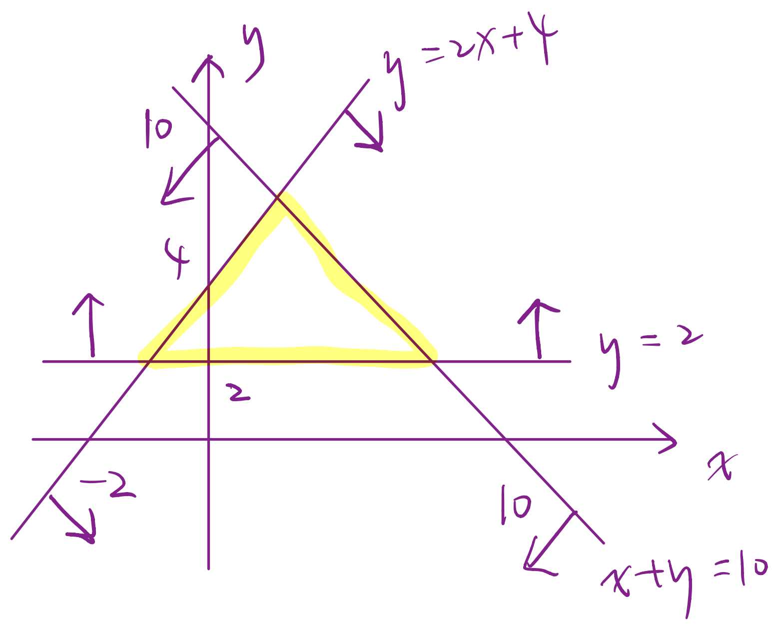
(d) $\begin{cases} x-y+1<0 \\ x>-1 \\ y\le 3 \end{cases}$

斜線 (oblique).
用 x-int, y-int 畫圖
y=0 x=0



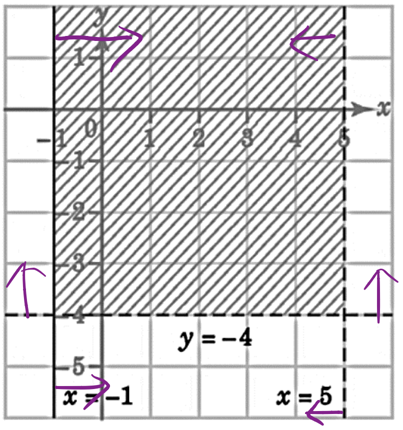
(e) $\begin{cases} x+y\le 10 \\ y\le 2x+4 \\ y\ge 2 \end{cases}$

x-int=10, y-int = 10
x int = -2, y-int = 4



5. In each of the following, find the system of inequalities whose solutions are represented by the shaded region in the figure.

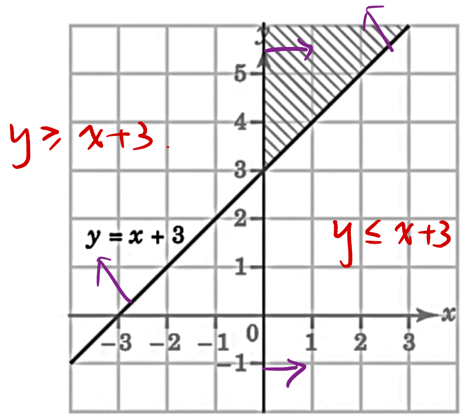
(a)



The required system of inequalities:

$$\begin{cases} y > -4 \\ x \geq -1 \\ x < 5 \end{cases}$$

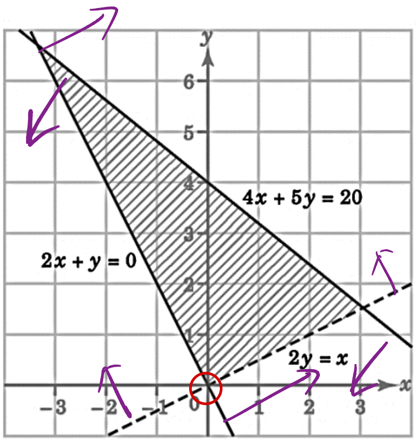
(b)



The required system of inequalities:

$$\begin{cases} y \geq x+3 \\ x \geq 0 \end{cases}$$

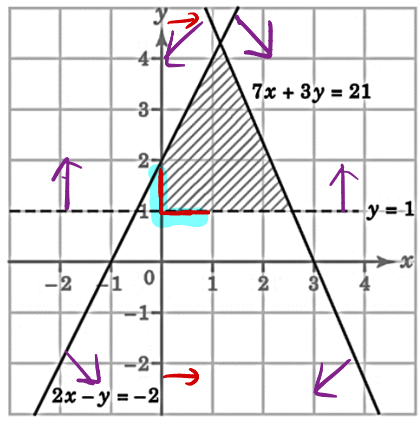
(c)



The required system of inequalities:

$$\begin{cases} 2x+y \geq 0 \\ 4x+5y \leq 20 \\ 2y > x \end{cases}$$

(d)



The required system of inequalities:

$$\begin{cases} y > 1 \\ 2x-y \geq -2 \\ 7x+3y \leq 21 \\ x \geq 0 \end{cases}$$

19.3 Introduction to Linear Programming

A. Linear Functions in Two Variables

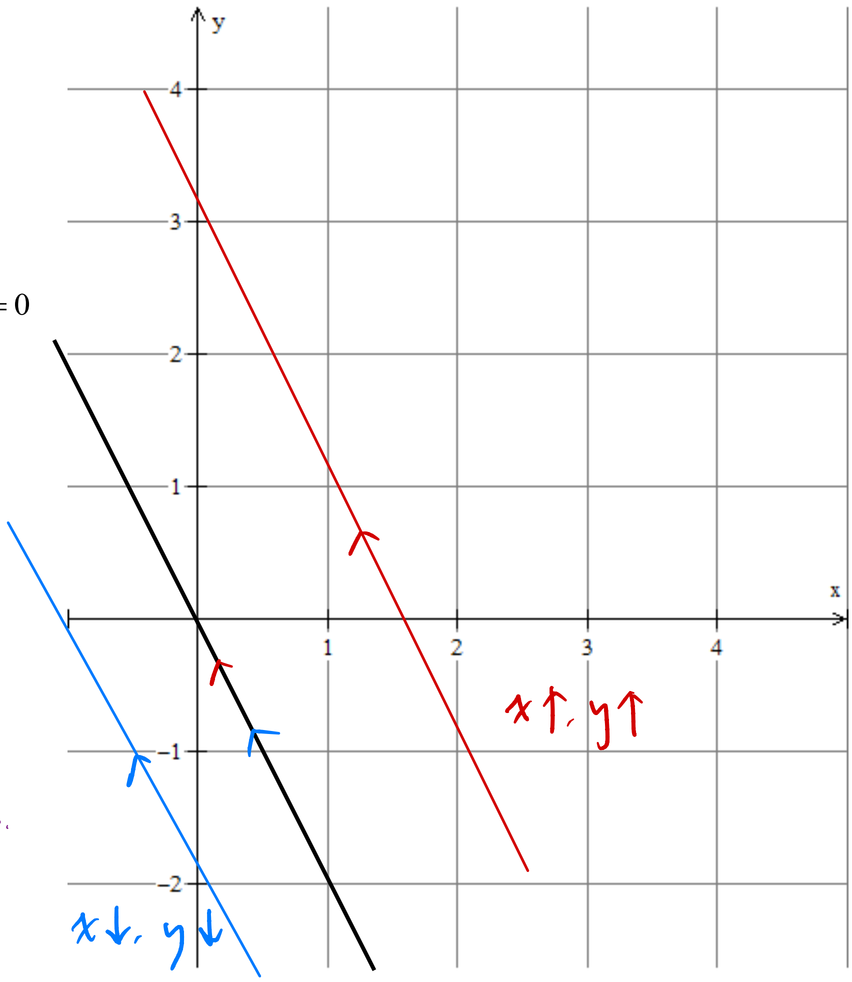
Consider the following function.

$$P = 2x + y$$

The value of the variable P depends on both values of x and y and the degree of x and y are both 1. This kind of function is called a lin**ear function in two variables x and y**. In general, linear functions in two variables are in the form $P = ax + by + c$.

To represent the linear function in two variables x and y graphically, we can draw the graph of $P = 2x + y$ for different values of P. Suppose the graph of $2x + y = 0$ is drawn.

$P = 2x + y = 0$



$y = -2x + k$

$2x + y = k$

parallel lines.

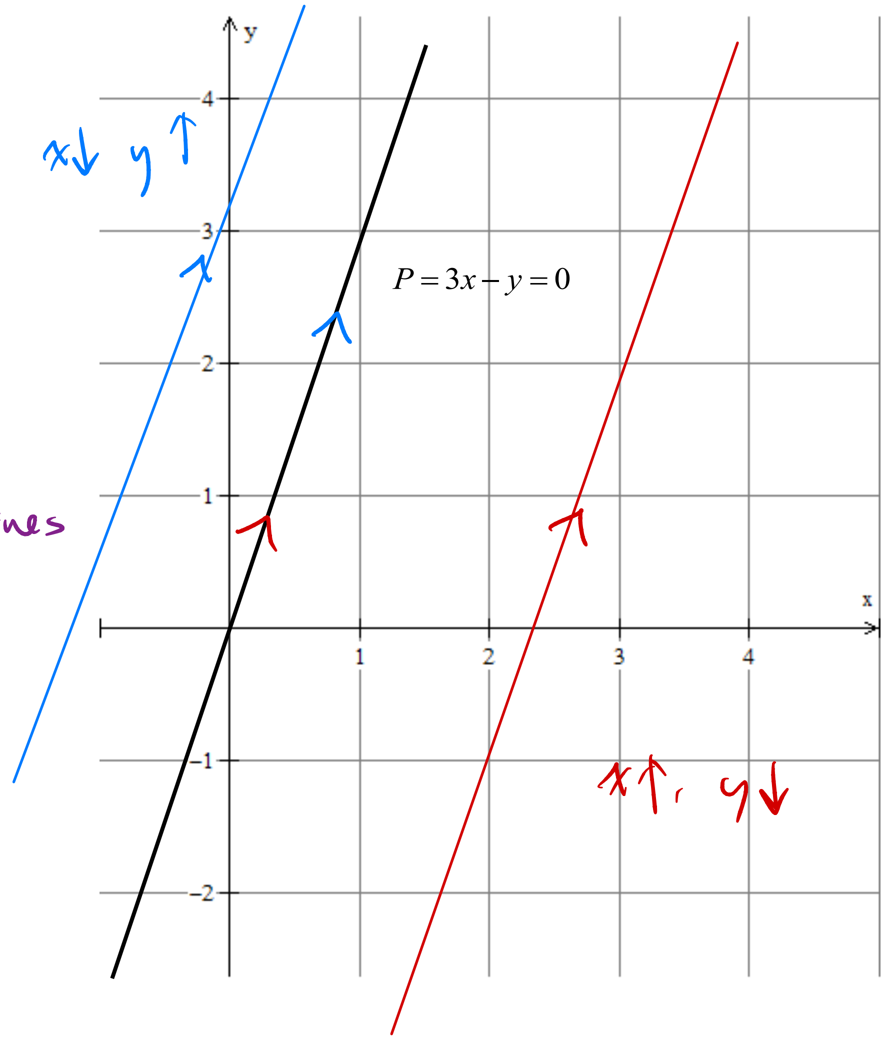
How does the value of $2x + y$ change as the graph of $P = 2x + y$ is

(i) translated to the right? P increases / decreases

(ii) translated to the left? P increases / decreases

Consider another linear function in two variables x and y $P = 3x - y$.

Suppose the graph of $3x - y = 0$ is drawn.



$y = 3x - k$

$3x - y = k$

parallel lines

How does the value of $3x - y$ change as the graph of $P = 3x - y$ is

(i) translated to the right? P increases / decreases

(ii) translated to the left? P increases / decreases

In general, for a linear function $P = ax + by + c$,

	$a > 0$	$a < 0$
Translate the line $ax + by + c = 0$ **rightwards**	Value of P increases	Value of P decreases
Translate the line $ax + by + c = 0$ **leftwards**	Value of P decreases	Value of P increases

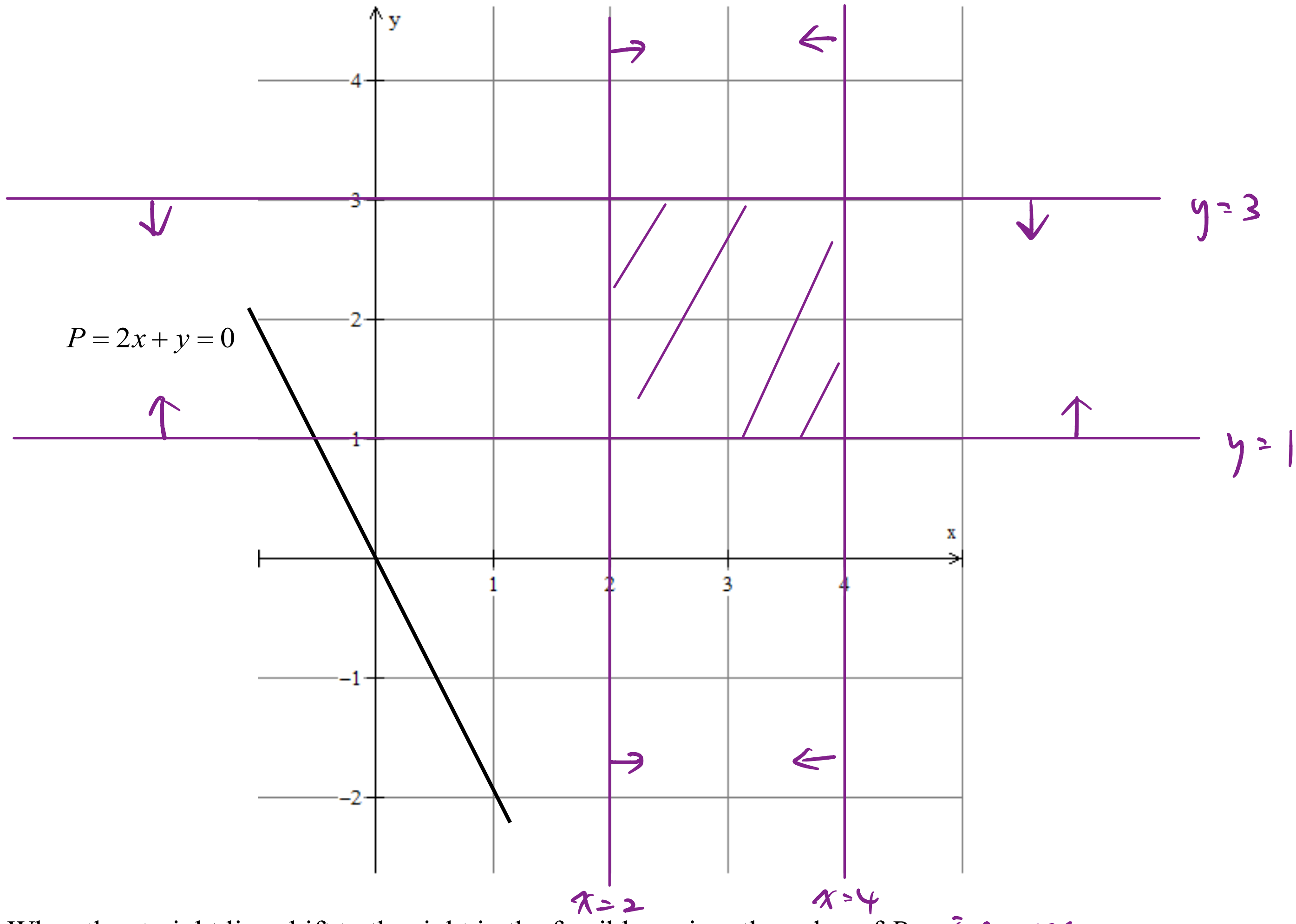
B. Linear Programming

Linear Programming is the study of finding **optimal solutions under some constraints.**
For example, $P = 2x + y$ is a linear function in x and y. If the values of x and y are restricted, then the value of P will also be restricted. Each restriction on the values of x and y is called a **constraint.**

$\begin{cases} 2 \le x \le 4 \\ 1 \le y \le 3 \end{cases}$

Constraints	***Maximum and minimum value of P*** ($P = 2x + y$)
$\begin{cases} x \le 4 \\ y \le 3 \\ x \ge 2 \\ y \ge 1 \end{cases}$	max. → $x\uparrow$, $y\uparrow$ when $x = 4$, $y = 3$ max. $P = 2\times 4 + 3 = 11$ min. → $x\downarrow$, $y\downarrow$ when $x = 2$, $y = 1$ min. $P = 2\times 2 + 1 = 5$

Consider the graph of $P = 2x + y = 0$ again. Draw the feasible region of the above constraints.



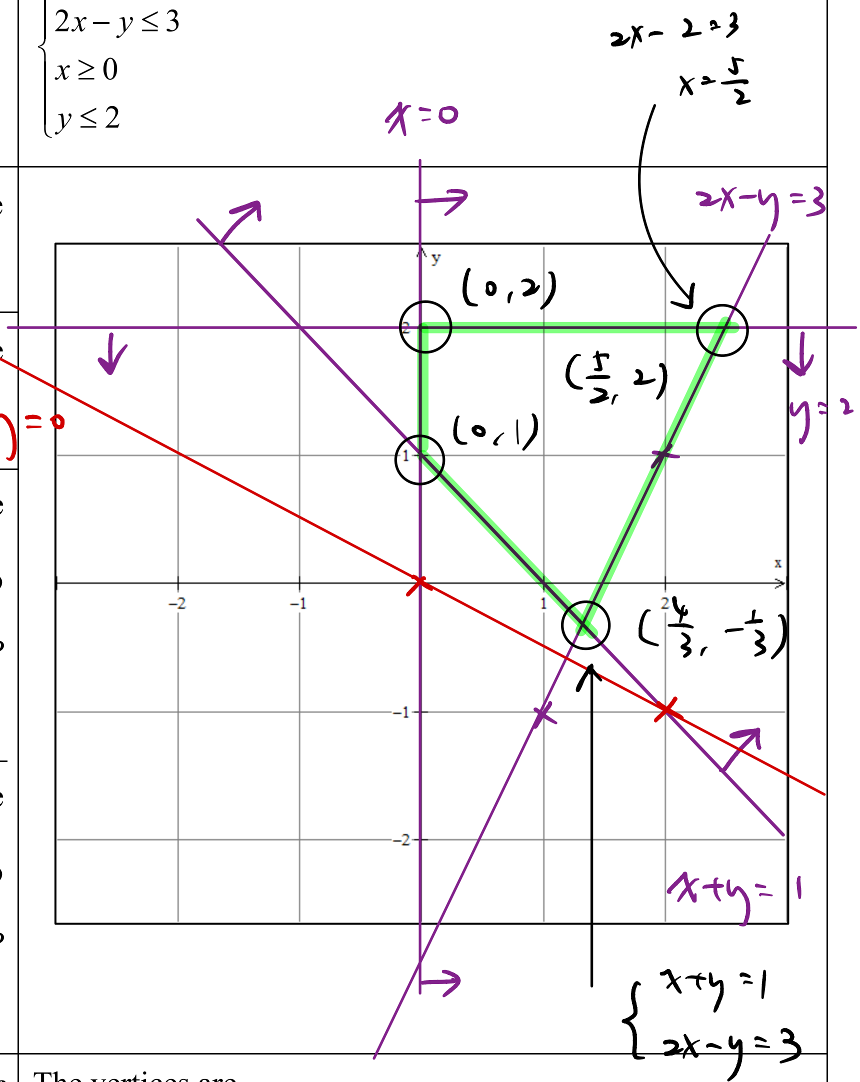
When the straight line shift to the right in the feasible region, the value of P increases.

P attains maximum when the line shift to the rightmost position in the feasible region.

When the straight line shift to the left in the feasible region, the value of P decreases.

P attains minimum when the line shift to the leftmost position in the feasible region.

$P(x, y)$

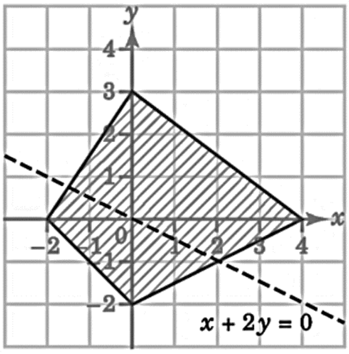
Steps	Example: Find the maximum and the minimum values of the function $P = x + 2y$ subject to the following constraints: $\begin{cases} x+y \ge 1 \\ 2x - y \le 3 \\ x \ge 0 \\ y \le 2 \end{cases}$
1. Draw and shade the region that satisfies the constraints.	
2. On the same coordinate plane, draw the line $x + 2y = 0$. $x+2y=k$ $x+2y=0$	
3. Translate the line $x + 2y = 0$ to the right within the feasible region to find the ordered pair (x, y) at which P attains its maximum.	
4. Translate the line $x + 2y = 0$ to the left. within the feasible region to find the ordered pair (x, y) at which P attains its minimum.	
5. We can test the values of P at all **vertices of the feasible region.**	The vertices are $P(0,1) = 0 + 2\cdot 1 = 2$ $P(0,2) = 0 + 2\cdot 2 = 4$ $P(\frac{4}{3}, -\frac{2}{3}) = \frac{4}{3} + 2(-\frac{2}{3}) = \frac{2}{3}$ $P(\frac{5}{2}, 2) = \frac{5}{2} + 2\cdot 2 = \frac{13}{2}$ max. $P = \frac{13}{2}$, min. $P = \frac{2}{3}$

Example

6. In each of the following, find the maximum and minimum values of the linear function P in the shaded region.

(a) $P = x + 2y$

右，P↑
左，P↓

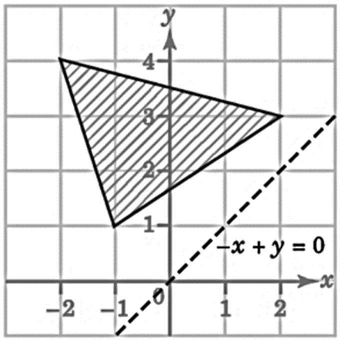


P(0,3) = 6
P(4,0) = 4
P(−2,0) = −2
P(0,−2) = −4

max. P = 6
min P = −4

(b) $P = -x + y$

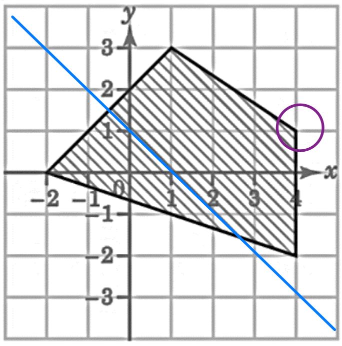
右，P↓
左，P↑



P(−1,1) = 2
P(−2,4) = 6
P(2,3) = 1.

max. P = 6
min. P = 1

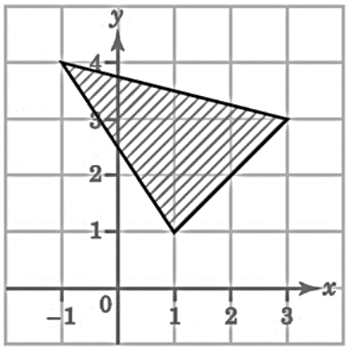
(c) $P = x + y$



x+y = 1

Add the line x+y = 1
Translate the line to the right within feasible region,
P attains max. at (4,1)
max. P = 4+1 = 5
Translate the line to the left within feasible region,
P attains min at (−2,0)
min P = −2+0 = −2

(d) $P = -3x - y$



P(−1,4) = −3(−1) − 4 = −1
P(1,1) = −3 −1 = −4
P(3,3) = −3·3 − 3 = −12

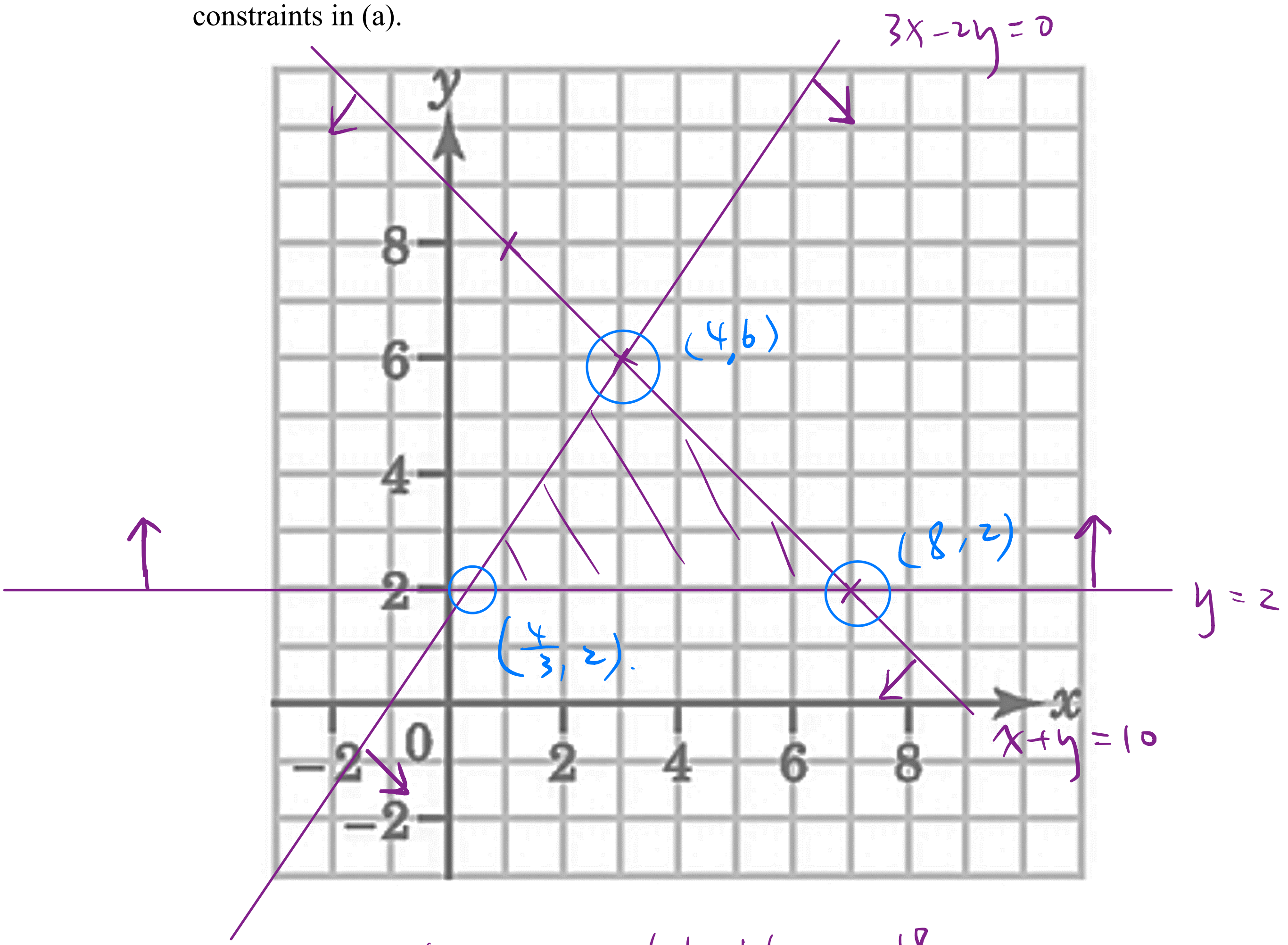
max. P = −1
min P. = −12

Example

7. (a) Draw and shade the region that satisfies the following constraints:

$$\begin{cases} x+y \le 10 \\ 3x-2y \ge 0 \\ y \ge 2 \end{cases}$$

(b) Find the maximum and the minimum values of the function $Q=-6x+y$ subject to the constraints in (a).



$Q(4,6) = -6\cdot 4+6 = -18$

$Q(\frac{4}{3}, 2) = -6\cdot\frac{4}{3}+2 = -6$

$Q(8,2) = -6\cdot 8+2 = -46$

$\therefore$ max. $Q = -6$,

min $Q = -46$.

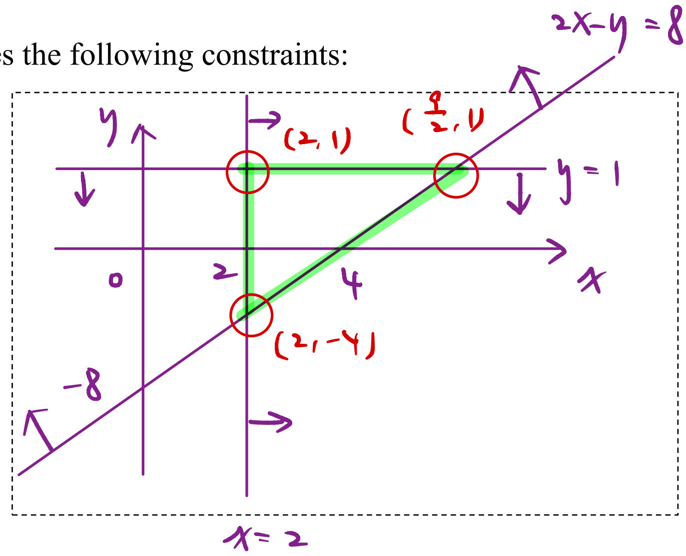
8. (a) Draw and shade the region that satisfies the following constraints:

$$\begin{cases} 2x-y\le 8 \\ x\ge 2 \\ y\le 1 \end{cases}$$

$2x-y=8$

x-int $=4$

y-int $=-8$



(b) Find the maximum and the minimum values of the function $P=3x+y$ subject to the constraints in (a).

$P(2,-4) = 6-4 = 2$ ← min.

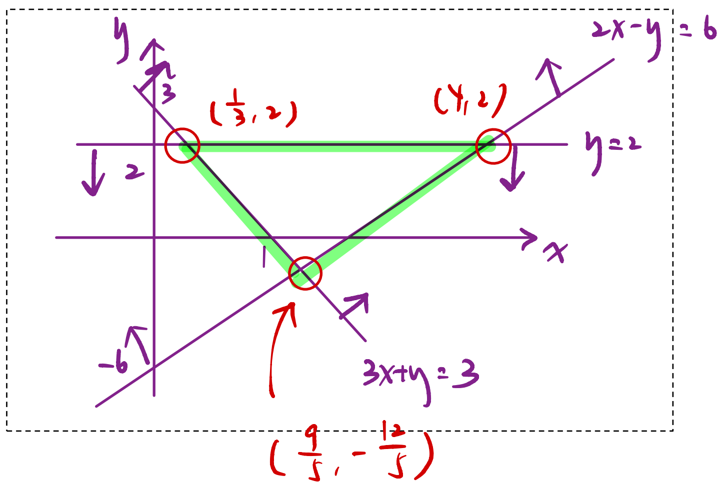
$P(2,1) = 6+1 = 7$

$P(\frac{9}{2},1) = \frac{27}{2}+1 = \frac{29}{2}$ ← max.

9. (a) Draw and shade the region that satisfies the following constraints:

$$\begin{cases} 3x+y\ge 3 \\ 2x-y\le 6 \\ y\le 2 \end{cases}$$

	x-int.	y-int.
$3x+y=3$	1	3
$2x-y=6$	3	−6



(b) Find the maximum and the minimum values of the function $P=2x-3y-6$ subject to the constraints in (a).

$P(\frac{1}{3},2) = \frac{2}{3}-6-6 = \frac{-34}{3}$ ← min.

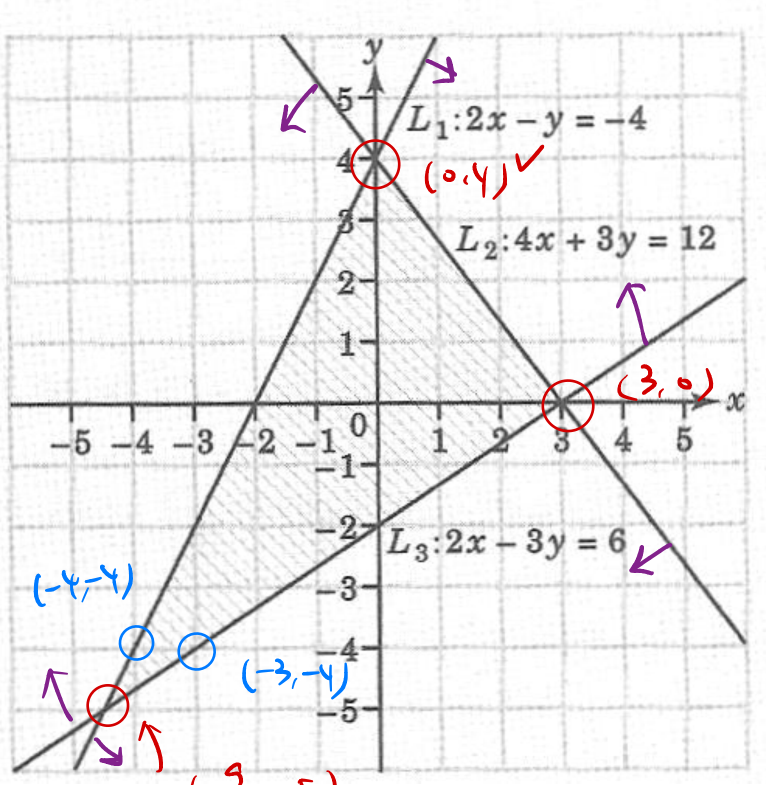
$P(4,2) = 8-6-6 = -4$

$P(\frac{9}{5},-\frac{12}{5}) = \frac{18}{5}+\frac{36}{5}-6 = \frac{24}{5}$ ← max.

10. In the figure, the shaded region represents the solutions of a system of inequalities.

(a) Write down the system of inequalities.

$$\begin{cases} 2x - y \geq -4 \\ 4x + 3y \leq 12 \\ 2x - 3y \leq 6 \end{cases}$$



x, y - integers

(b) If **(*x*, *y*) is an integral point in the shaded region,** find the maximum and the minimum value of $B = x + y$ subject to the constraints in (a).

$B(0, 4) = 4 \leftarrow$ max.

$B(3, 0) = 3$

$B(-4, -4) = -8 \leftarrow$ min.

$B(-3, -4) = -7$

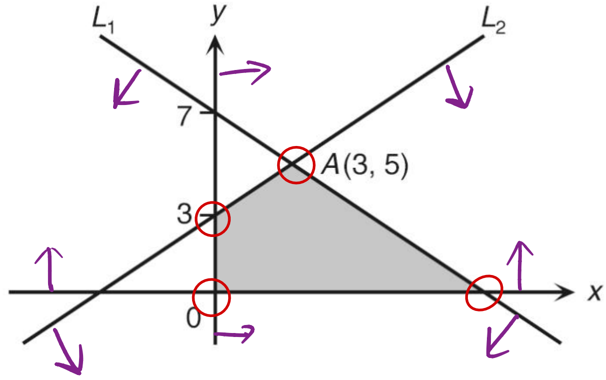
11. In the figure, two straight lines L_1 and L_2 intersect at $A(3, 5)$. The y-intercepts of L_1 and L_2 are 7 and 3 respectively.

(a) (i) Find the equations of L_1 and L_2.

(ii) Find the x-intercept of L_1.

(b) Write down the system of inequalities whose solutions are represented by the shaded region.

(c) Find the maximum and the minimum values of $P = 3x - 2y$ subject to the constraints obtained in (b).



(a) (i) L_1: $\dfrac{y-5}{x-3} = \dfrac{7-5}{0-3}$

$-3y + 15 = 2x - 6$

$2x + 3y - 21 = 0$

L_2: $\dfrac{y-5}{x-3} = \dfrac{3-5}{0-3}$

$-3y + 15 = -2x + 6$

$2x - 3y + 9 = 0$

(ii) put $y=0$ into L_1, $x = \dfrac{21}{2}$, x-int $= \dfrac{21}{2}$

(b) $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + 3y - 21 \leq 0 \\ 2x - 3y + 9 \geq 0 \end{cases}$

(c) $P(0, 0) = 0$

$P(0, 3) = -6$

$P(\frac{21}{2}, 0) = \dfrac{63}{2}$

$P(3, 5) = -1$

max. $P = \dfrac{63}{2}$

min. $P = -6$

12. The straight lines L_1 and L_2 intersect at (6, 7). The y-intercept of L_1 is 4 while the slope of L_2 is $\frac{5}{3}$. Let R be the region (including the boundary) bounded by L_1, L_2 and the y-axis.

(a) Find the equation of L_1 and L_2.

(b) It is given that R represents the solutions of a system of inequalities. Write down the system of inequalities.

(c) Find the maximum and minimum values of $3x+2y-19$, where (x, y) is a point lying on R.

(a) L_1: $\frac{y-7}{x-6} = \frac{7-4}{6-0}$

$2y-14 = x-6$

$x-2y+8=0$

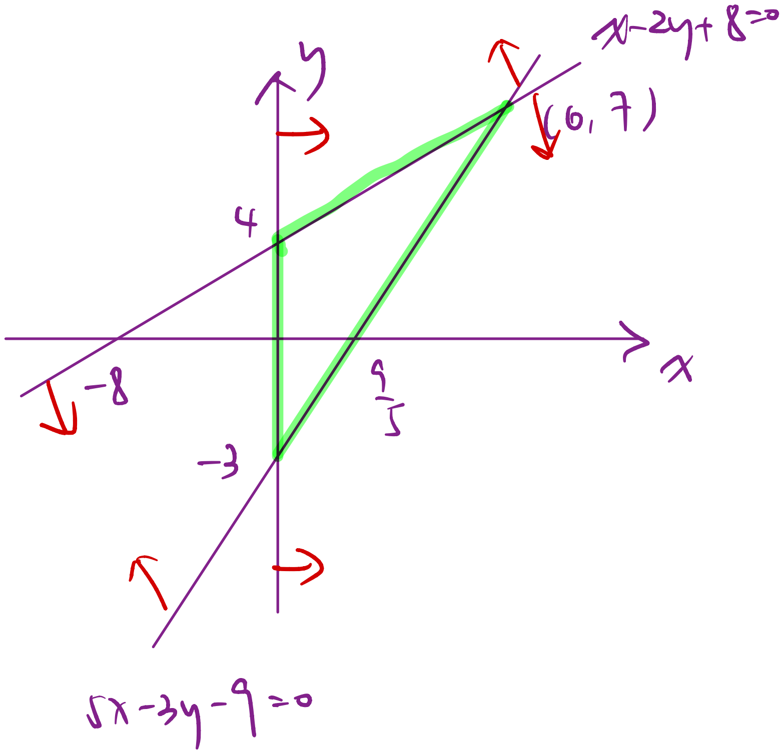
L_2: $\frac{y-7}{x-6} = \frac{5}{3}$

$3y-21 = 5x-30$

$5x-3y-9=0$

	x-int.	y-int.
$x-2y+8=0$	-8	4
$5x-3y-9=0$	$\frac{9}{5}$	-3

(b) $\begin{cases} x-2y+8 \geq 0 \\ 5x-3y-9 \leq 0 \\ x \geq 0 \end{cases}$



(c) $P = 3x+2y-19$

$P(0, 4) = -11$

$P(0, -3) = -25$ ← min

$P(6, 7) = 13$ ← max.

13. The equation of the straight line L_1 is $7x-8y+95=0$. The straight lines L_1 and L_2 are perpendicular to each other. The x-intercept of L_2 is $-\frac{11}{2}$, while the slope and the y-intercept of the straight line L_3 are $\frac{1}{2}$ and 10 respectively. Let D be the region (including the boundary) bounded by L_1, L_2, L_3, the x-axis and the y-axis.

(a) It is given that D represents the solutions of a system of inequalities. Find the system of inequalities.

(b) Suppose (x, y) is a point lying on D. Find the constant k such that the maximum value of $-5x+6y+k$ is 98.

(a) $m_{L_1} = \frac{7}{8}$

$m_{L_2} = \frac{-8}{7}$

L_2: $\frac{y-0}{x+\frac{11}{2}} = \frac{-8}{7}$

$7y = -8x - 44$

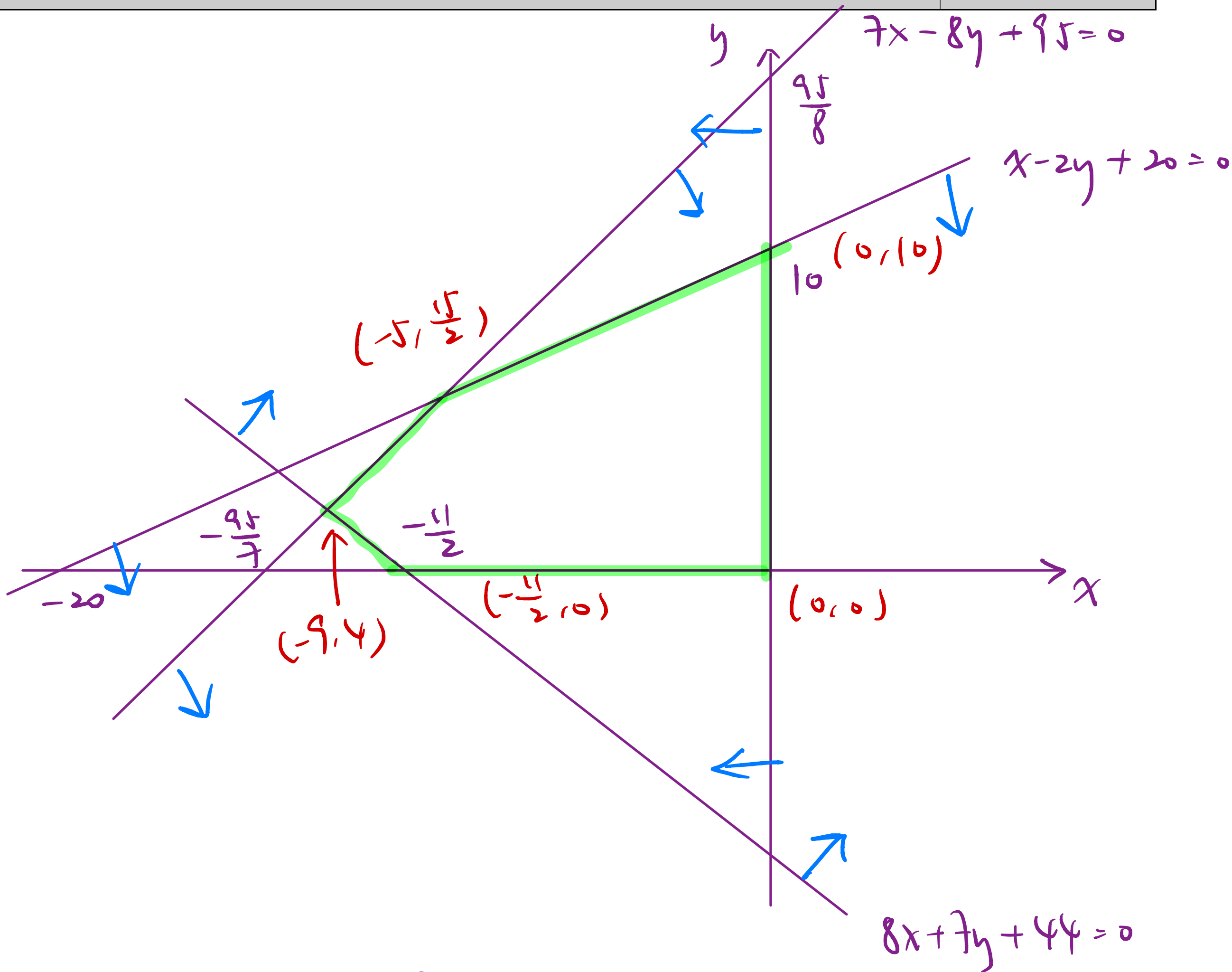
$8x + 7y + 44 = 0$

L_3: $y = \frac{1}{2}x + 10$

$2y = x + 20$

$x - 2y + 20 = 0$

	x-int.	y-int.
$7x-8y+95=0$	$-\frac{95}{7}$	$\frac{95}{8}$
$8x+7y+44=0$	$-\frac{11}{2}$	$-\frac{44}{7}$
$x-2y+20=0$	-20	10



(b)
$$\begin{cases} x-2y+20\geq 0 \\ 7x-8y+95\geq 0 \\ 8x+7y+44\geq 0 \\ x\leq 0 \\ y\geq 0 \end{cases}$$

(c) $P(x,y) = -5x+6y+k$

$P(0,0)=k$

$P(0,10)=60+k$

$P(-\frac{11}{2},0)=\frac{55}{2}+k$

$P(-5,\frac{15}{2}) = 70+k$

$P(-9,4) = 69+k$

$\therefore$ max. $P = 70+k = 98$

$k = 28$

19.4 Applications of Linear Programming

Linear programming helps us solve problems involving maximization or minimization of some quantities under certain constraints. For example, it is used in many sectors to maximize the profit or minimize the cost with limited resources.

In general, we can follow the procedure below to solve these problems.

non-negative ≠ positive (include 0)

1. Identify the unknown quantities and represent them by letters, say x and y.
2. Identify all constraints and express them as inequalities in terms of x and y.
   Remark: In real life situation, x and y are usually restricted to ***non-negative real numbers*** *or* ***integers.***
3. Draw and shade the region that satisfies all the constraints on a coordinate plane.
4. Express the objective function in terms of x and y.
5. Find the maximum (or minimum) value of the objective function subject to the constraints.

Example

14. A certain food company produces x kg of chocolate cake and y kg of lemon cake every day. The cost of producing 1 kg of chocolate cake and that of lemon cake are \$50 and \$100 respectively. The company produces the cakes under the following conditions.

> **1. At most \$2000 is used to produce these two kinds of cake every day.**
> **2. The total weight of these two kinds of cake produced per day should not exceed 30 kg.**

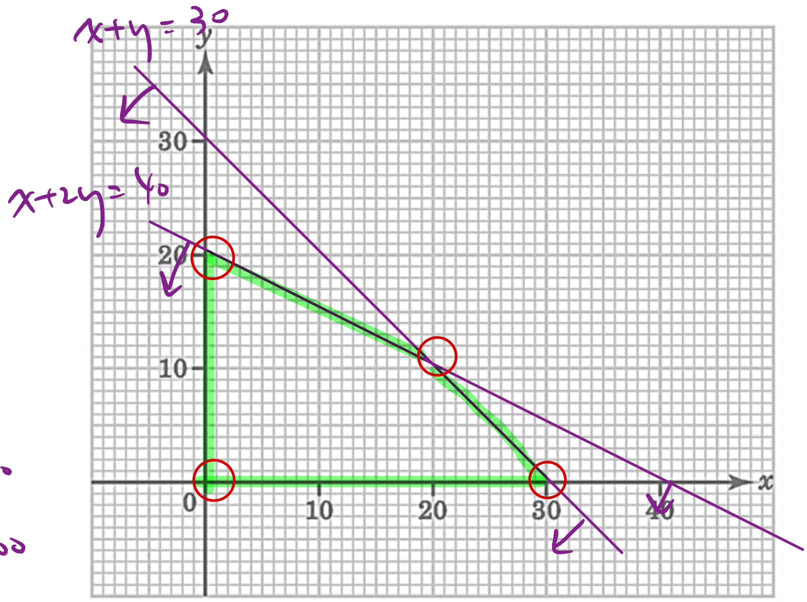
(a) Write down the constraints for x and y.

(b) Draw and shade the feasible region of the constraints obtained in (a).

(c) The profits made from selling chocolate cake and lemon cake are \$80/kg and \$120/kg respectively. Find the weights of the two kinds of cake should be produced per day in order to obtain the maximum profit. What is the maximum profit?

$x + 2y \le 40$

(a) $\begin{cases} 50x + 100y \le 2000 \\ x + y \le 30 \\ x \ge 0 \\ y \ge 0 \end{cases}$



(c) $P = 80x + 120y$

$P(0,0) = 0$ $\quad P(30,0) = 2400$

$P(0,20) = 2400$ $\quad P(20,10) = 2800$

$\therefore$ max. profit $= \$2800$

15. The person-in-charge of a travel tour hires two types of coaches A and B to carry 75 tourists and 120 suitcases.

    Each coach A can carry 15 passengers and 40 suitcases. Each coach B can carry 45 passengers and 60 suitcases.

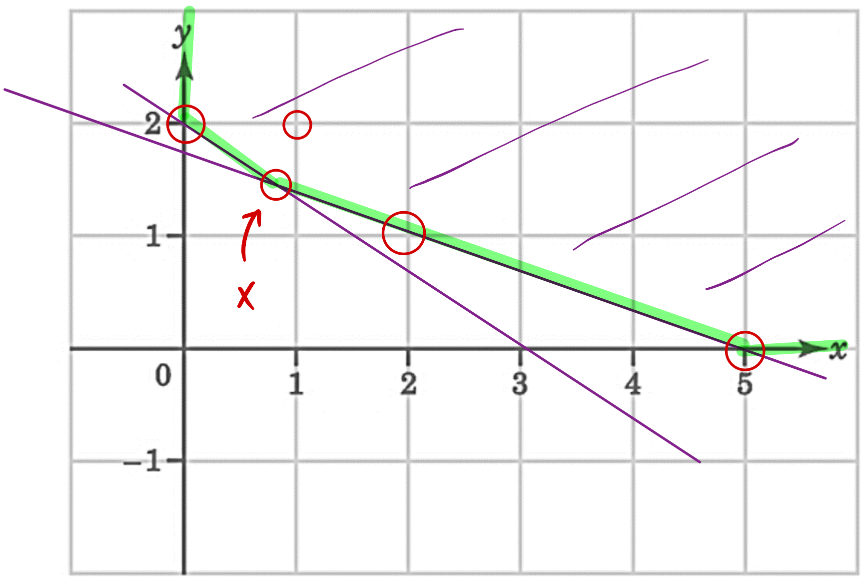
    Suppose the person-in-charge hires x coaches A and y coaches B.

    (a) Write down all the constraints on x and y.

    (b) Draw and shade the region that satisfies the constraints in (a) on a coordinate plane.

    (c) The costs of hiring each coach A and each coach B are \$800 and \$2000 respectively. Someone claims that the cost of hiring coaches can be less than \$3500 if all the tourists and suitcases are carried. Do you agree? Explain your answer.

(a) $\begin{cases} x, y \text{ are non-negative integers} \\ 15x + 45y \geq 75 \\ 40x + 60y \geq 120 \end{cases}$



(c) $C = 800x + 2000y$

$C(0,2) = 4000$

$C(1,2) = 4800$

$C(2,1) = 3600$

$C(5,0) = 4000$

$\therefore$ min. cost $= \$3600$, $> \$3500$

$\therefore$ cost cannot be less than \$3500

$\therefore$ the claim is disagreed.

16. A carpenter wants to make x tables and y chairs in a month. The amounts of materials and manpower required for making a table and a chair are shown below.

	Wood *plank*	Manpower
Table	6 units	4 man-days
Chair	2 units	2 man-days

Suppose the carpenter has 24 units of wood planks and he can work at most 20 man-days a month.

(a) Write down all the constraints on x and y.

(b) Indicate the solutions that satisfies the constraints in (a) on a coordinate plane.

(c) Assume all the tables and chairs can be sold. The profits from selling a table and a chair are \$2000 and \$800 respectively. How many tables and chairs should be made in order to maximize the profit? What is the maximum profit?

(a) $\begin{cases} x, y \text{ are non-negative integers} \\ 6x + 2y \le 24 \\ 4x + 2y \le 20 \end{cases}$

(c) $P = 2000x + 800y$

$P(0, 10) = 8000$

$P(4, 0) = 8000$

$P(2, 6) = 8800$

$P(0, 0) = 0$

$\therefore$ max. profit = \$8800

$\therefore$ 2 tables and 6 chairs should be made.