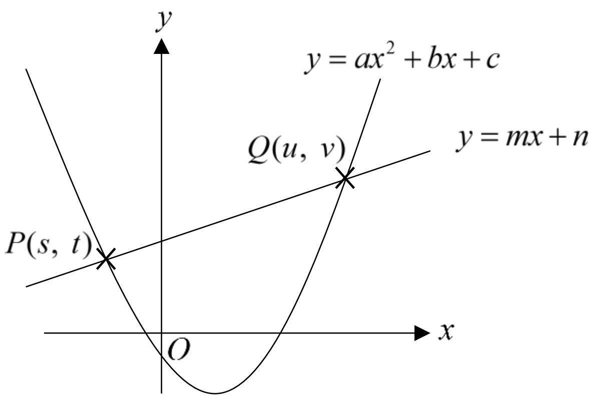
Chapter 9 More About Equations
Supplementary Notes

Name: ________________________________ (   )   Class: F.4 _____

9.1 Solving Simultaneous Equations by the Graphical Method



Draw the graphs of the two equations in the same rectangular coordinate plane. The coordinates of the points of intersection P and Q can give the solutions which are **approximate values**.

i.e. $(x, y) =$ (s , t) or (u , v)

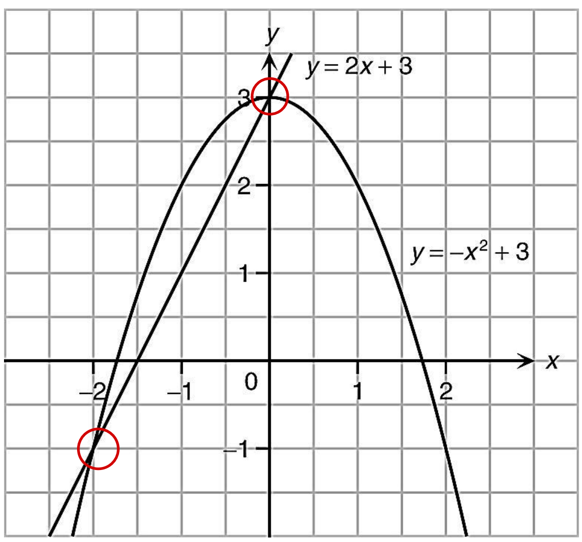
Coordinates of the intersection(s) of the graphs

⇕

Solution(s) of simultaneous equations

Example

1. The figure shows the graphs of $y = -x^2 + 3$ and $y = 2x + 3$, find the solutions of the simultaneous equations $\begin{cases} y = -x^2 + 3 \\ y = 2x + 3 \end{cases}$ using the graph.



The coordinates of the points of intersection of two graphs give the solutions of the simultaneous equations.

$x = 0, -2$

$y = 3, -1$

X

① $(x, y) = (0, 3)$ or $(-2, -1)$

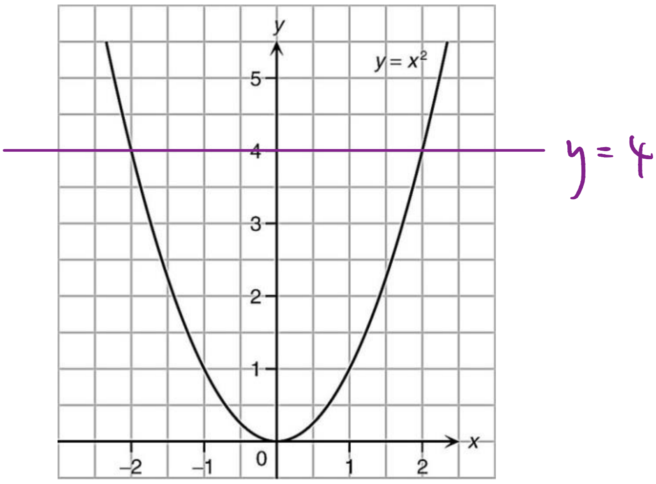
② $\begin{cases} x = 0 \\ y = 3 \end{cases}$ or $\begin{cases} x = -2 \\ y = -1 \end{cases}$

③ When $x = 0$, $y = 3$

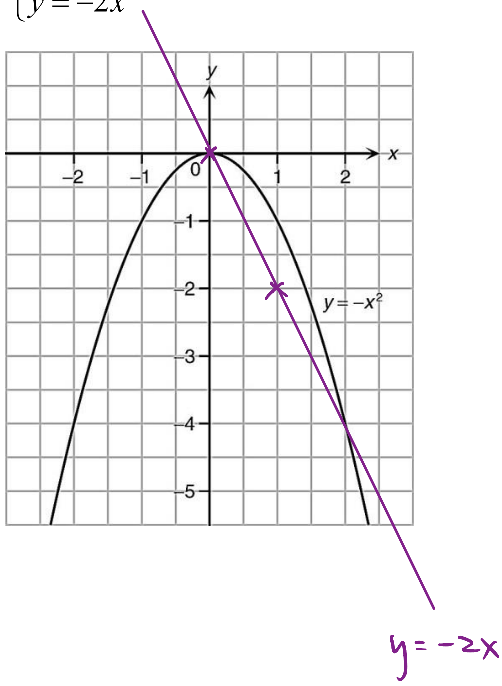
$x = -2$, $y = -1$

2. In each of the following, solve the given simultaneous equations by adding a suitable straight line to the figure.

(a) $\begin{cases} y = x^2 \\ y = 4 \end{cases}$ (2, 4) or (−2, 4)



(b) $\begin{cases} y = -x^2 \\ y = -2x \end{cases}$ (0, 0) or (2, −4)



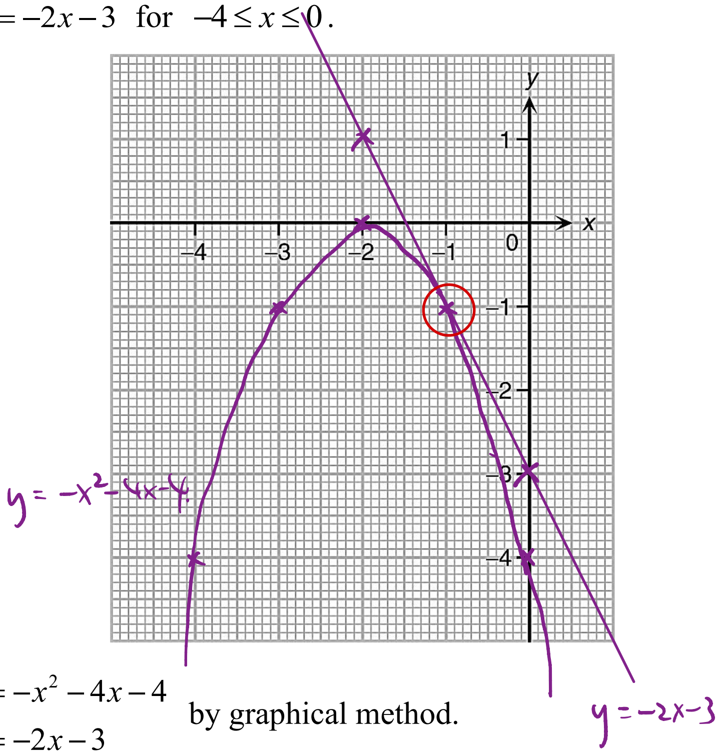
3. (a) Draw the graphs of $y = -x^2 - 4x - 4$ and $y = -2x - 3$ for $-4 \le x \le 0$.

$y = -x^2 - 4x - 4$

x	−4	−3	−2	−1	0
y	−4	−1	0	−1	−4

$y = -2x - 3$

x	−2	0
y	1	−3



(b) Hence solve the simultaneous equations $\begin{cases} y = -x^2 - 4x - 4 \\ y = -2x - 3 \end{cases}$ by graphical method.

(−1, −1)

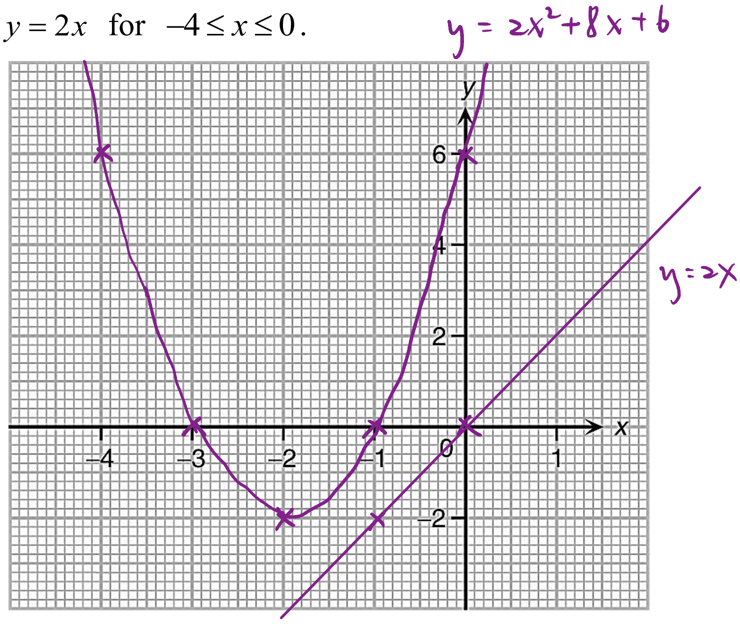
4. (a) Draw the graphs of $y = 2x^2 + 8x + 6$ and $y = 2x$ for $-4 \le x \le 0$.

$y = 2x^2 + 8x + 6$

x	–4	–3	–2	–1	0
y	6	0	–2	0	6

$y = 2x$

x	–1	0	1
y	–2	0	2



(b) Hence solve the simultaneous equations $\begin{cases} y = 2x^2 + 8x + 6 \\ y = 2x \end{cases}$ by graphical method.

No real solution

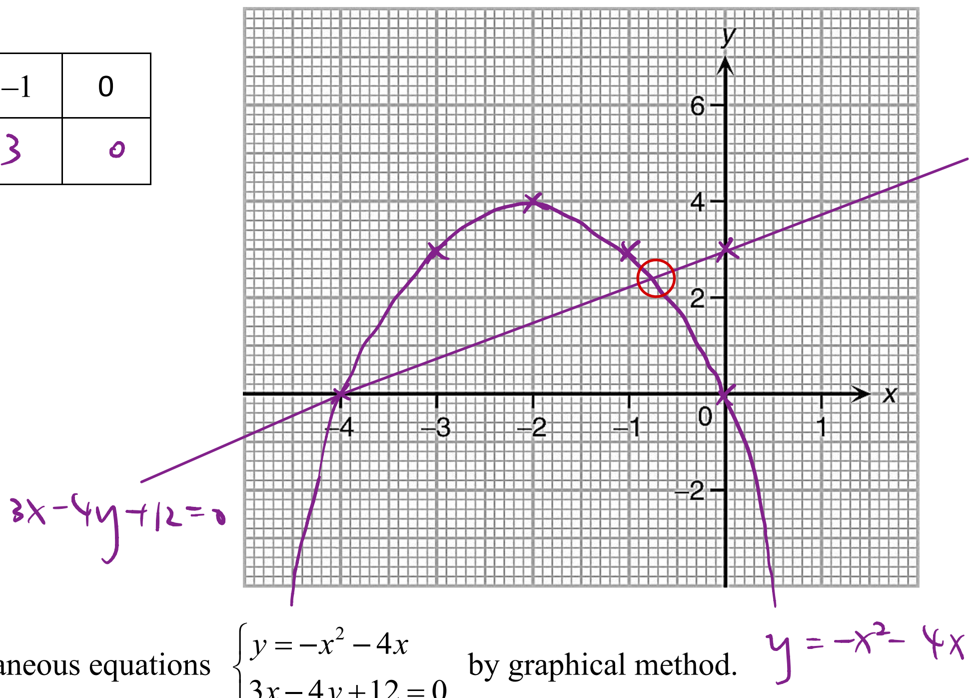
5. (a) Draw the graphs of $y = -x^2 - 4x$ and $3x - 4y + 12 = 0$ for $-4 \le x \le 0$.

$y = -x^2 - 4x$

x	–4	–3	–2	–1	0
y	0	3	4	3	0

$3x - 4y + 12 = 0$

x	–4	0
y	0	3



(b) Hence solve the simultaneous equations $\begin{cases} y = -x^2 - 4x \\ 3x - 4y + 12 = 0 \end{cases}$ by graphical method.

(–4, 0) or (–0.8, 2.4) ← approximate value

corr. to nearest 0.2

corr. to nearest 0.1

9.2 Solving Simultaneous Equations by the Algebraic Method

In this section, we are going to solve a pair of simultanoeous equations in two unknowns, in which one equation is linear and the other equation is quadratic.

$$\text{Solve } \begin{cases} y = ax^2 + bx + c & \longleftarrow \text{Quadratic Equation} \\ y = mx + n & \longleftarrow \text{Linear Equation} \end{cases}$$

The algebraic method can give the **exact** solutions of the equations. The most commonly used algebraic method is the **method of substitution**. ← form an equation with one unknown.

Steps	Example: Solve $\begin{cases} y = x^2 - 5x + 4 \\ y + x - 1 = 0 \end{cases}$
1. Change one of the variable ~~(usually y)~~ of the **linear equation** as the subject.	$\begin{cases} y = x^2 - 5x + 4 \ldots (i) \\ y + x - 1 = 0 \ldots\ldots (ii) \end{cases}$ From *(ii)*, $y = 1 - x \ldots (iii)$
2. Substitute the subject (y) into the quadratic equation to eliminate the variable (y).	Put *(iii)* into *(i)* $y = x^2 - 5x + 4$ $1 - x = x^2 - 5x + 4$ $x^2 - 4x + 3 = 0$
3. Solve the quadratic equation in one unknown x.	Solve $x^2 - 4x + 3 = 0$ $(x-3)(x-1) = 0$ $x = 3$ or $x = 1$
4. Substitute the value of x obtained in step 3 into the equation formed in step 1 to solve for another variable y.	By substituting $x = 3$ into (*iii*), $y = 1 - 3 = -2$ By substituting $x = 1$ into (*iii*), $y = 1 - 1 = 0$
5. Write down the different sets of solution of the simultaneous equations.	$(x, y) = (3, -2)$ or $(1, 0)$ Note: The solutions can also be expressed as: $\begin{cases} x = 3 \\ y = -2 \end{cases}$ or $\begin{cases} x = 1 \\ y = 0 \end{cases}$

Example

6. Solve the following simultaneous equations.

(a) $\begin{cases} x^2 - y + 1 = 0 & \cdots (1) \\ y = 3 - x & \cdots (2) \end{cases}$

put (2) into (1),

$x^2 - (3-x) + 1 = 0$

$x^2 + x - 2 = 0$

$x = -2$ or 1

when $x = -2$, $y = 3-(-2) = 5$

$x = 1$, $y = 3 - 1 = 2$

(b) $\begin{cases} y = x^2 - 5x + 4 & \cdots (1) \\ y = 1 - x & \cdots (2) \end{cases}$

put (2) into (1)

$1 - x = x^2 - 5x + 4$

$x^2 - 4x + 3 = 0$

$x = 3$ or 1

when $x = 3$, $y = -2$

$x = 1$, $y = 0$

$x = \frac{3-y}{3}$ ✗ avoid fractions

(c) $\begin{cases} 3x + y = 3 \\ y + 5 = x^2 - x \end{cases}$

$y = 3 - 3x$

$3 - 3x + 5 = x^2 - x$

$x^2 + 2x - 8 = 0$

$x = -4$ or 2

when $x = -4$, $y = 15$

$x = 2$, $y = -3$

(d) $\begin{cases} x^2 - 2xy - 4x = 1 \\ x - y = 1 \end{cases}$

$x = 1 + y$

$(1+y)^2 - 2(1+y)\cdot y - 4(1+y) = 1$

$1 + 2y + y^2 - 2y - 2y^2 - 4 - 4y = 1$

$-y^2 - 4y - 4 = 0$

$y^2 + 4y + 4 = 0$

$y = -2$

$x = -1$

(e) $\begin{cases} 6x^2 - y^2 + 4 = 0 \\ y + 1 = 2x \end{cases}$

$y = 2x - 1$ ✓ / $x = \frac{y+1}{2}$ ✓✗

$y = 2x - 1$

$6x^2 - (2x-1)^2 + 4 = 0$

$6x^2 - (4x^2 - 4x + 1) + 4 = 0$

$2x^2 + 4x + 3 = 0$

$\Delta = 4^2 - 4 \cdot 2 \cdot 3 = -8, < 0$

$\therefore$ no real solution

(f) $\begin{cases} x^2 - 2y^2 = 8 \\ 2x - y + 4 = 0 \end{cases}$

$y = 2x + 4$

$x^2 - 2(2x+4)^2 - 8 = 0$

$x^2 - 2(4x^2 + 16x + 16) - 8 = 0$

$-7x^2 - 32x - 40 = 0$

$7x^2 + 32x + 40 = 0$

$\Delta = 32^2 - 4 \cdot 7 \cdot 40$

$= -96 < 0$

$\therefore$ no real solution.

(g) $x^2 + xy - 6 = 4x + 2y = 22$

$$\begin{cases} x^2 + xy - 6 = 22 & \cdots (1) \\ 4x + 2y = 22 & \cdots (2) \end{cases}$$

(2): $2x + y = 11$

$y = 11 - 2x$

$\therefore \quad x^2 + x(11-2x) - 6 - 22 = 0$

$x^2 + 11x - 2x^2 - 28 = 0$

$-x^2 + 11x - 28 = 0$

$x^2 - 11x + 28 = 0$

$x = 4$ or 7

when $x = 4$, $y = 3$

when $x = 7$, $y = -3$

(h) $x^2 - 2y^2 = 3x + 4y = 1$

$$\begin{cases} x^2 - 2y^2 = 1 \\ 3x + 4y = 1 \end{cases}$$

$3x + 4y = 1$

$y = \frac{1-3x}{4}$

$\therefore \quad x^2 - 2\left(\frac{1-3x}{4}\right)^2 = 1$

$x^2 - \frac{1}{8}(1-3x)^2 = 1$

$8x^2 - (1 - 6x + 9x^2) = 8$

$-x^2 + 6x - 9 = 0$

$x^2 - 6x + 9 = 0$

$x = 3$

$y = \frac{1 - 3 \cdot 3}{4} = -2$

Number of Intersections between a Straight Line and a Quadratic Curve

Consider the simultaneous equations $\begin{cases} y = ax^2 + bx + c & \text{.......(1)} \\ y = mx + n & \text{.......(2)} \end{cases}$

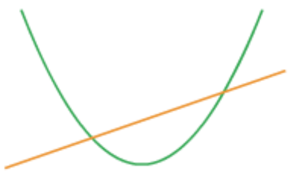
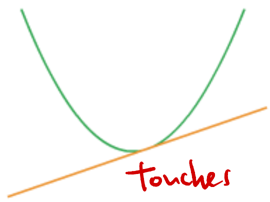
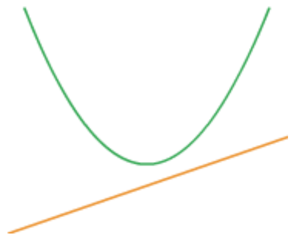
By substituting (2) into (1), we have

$$mx + n = ax^2 + bx + c$$

roots = x-coord. of intersection

$$ax^2 + (b-m)x + (c-n) = 0 \quad \text{.....(3)}$$

When we use graphical method to solve simultaneous equations in which one is linear and one is quadratic, there are 3 cases about the intersection:

	Case 1	Case 2	Case 3
	 Two points of intersection	 touches the curve One point of intersection	 No points of intersection
No. of real solutions	2 distinct real solutions	1 real solution	No real solutions
Discriminant (Δ) of $ax^2 + (b-m)x + (c-n) = 0$	$\Delta > 0$	$\Delta = 0$	$\Delta < 0$

<u>Example</u>

7. Let k be a constant. If the simultaneous equations $\begin{cases} y = x^2 - 2x - 1 \\ y = 4x - k \end{cases}$ have only one real solution, find k.

$x^2 - 2x - 1 = 4x - k$

$x^2 - 6x + k - 1 = 0$

$\Delta = 0$, $(-6)^2 - 4 \cdot 1 \cdot (k-1) = 0$

$36 - 4k + 4 = 0$

$k = 10$

8. Let k be a constant. The simultaneous equations $\begin{cases} y = 2x^2 + kx + 5 \\ x - y = k \end{cases}$ have only one real solution.

   (a) Find k.

   (b) Solve the simultaneous equations.

(a) $y = x - k$

$x - k = 2x^2 + kx + 5$

$0 = 2x^2 + (k-1)x + 5 + k$

$\Delta = 0$, $(k-1)^2 - 4 \cdot 2 \cdot (5+k) = 0$

$k^2 - 2k + 1 - 40 - 8k = 0$

$k^2 - 10k - 39 = 0$

$k = 13$ or -3

(b) when $k = 13$

$2x^2 + 12x + 18 = 0$

$x = -3$

$y = -3 - 13 = -16$

when $k = -3$

$2x^2 - 4x + 2 = 0$

$x = 1$

$y = 1 - (-3) = 4$

9. The quadratic curve $y = x^2 - 4x$ and the line $y = mx - 9$ intersect at only one point P. It is known that the line $y = mx - 9$ has a positive slope.

   (a) Find the value of m.

   (b) Find the coordinates of P.

(a) $x^2 - 4x = mx - 9$

$x^2 - 4x - mx + 9 = 0$

$x^2 - (m+4)x + 9 = 0$

$\Delta = 0$

$[-(m+4)]^2 - 4 \cdot 1 \cdot 9 = 0$

$(m+4)^2 = 36$

$m + 4 = 6$ or $m + 4 = -6$

$m = 2$ or $m = -10$ (rej.)

(b) $m = 2$

$x^2 - 6x + 9 = 0$

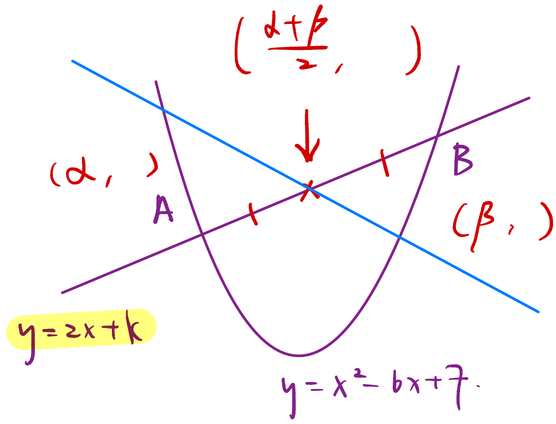
$x = 3$

$y = 2 \cdot 3 - 9 = -3$

$\therefore P = (3, -3)$

10. It is given that the straight line $y = 2x + k$ cuts the graph of $y = x^2 - 6x + 7$ at two distinct points A and B.
    (a) Express the coordinates of the mid-point of AB in terms of k.
    (b) If the straight line $x + y + 16 = 0$ bisects the line segment AB, find the value(s) of k.

> Recall:
>
> The x-coordinate of the mid-point = $\dfrac{\alpha+\beta}{2}$



(a) $2x+k = x^2-6x+7$

$0 = x^2 - 8x + 7 - k$

x coord of A, B

$\therefore$ x-coord of mid-point of AB

$= -\frac{-8}{1} \div 2$

$= 4$

$\therefore$ y-coord of mid-point of AB $= 2 \cdot 4 + k = 8 + k$

$\therefore$ mid-point of AB $= (4, 8+k)$

(b) the straight line $x+y+16=0$ passes through the mid-point of AB.

$\therefore$ $4 + 8 + k + 16 = 0$

$k = -28$

11. It is given that the line $y=4x-k$ cuts the graph of $y=-x^2+2x-3$ at two distinct points A and B.

(a) Express the coordinates of the mid-point of AB in terms of k.

(b) If the line $5x-2y=11$ bisects the line segment AB, find the value(s) of k.

(a) $4x-k=-x^2+2x-3$

$x^2+2x+3-k=0$

$\therefore$ x-coord. of mid-point of AB $= -\frac{2}{1} \div 2 = -1$

y-coord. of mid-point of AB $= 4(-1)-k$

$= -4-k$

$\therefore$ mid-point of AB $= (-1, -4-k)$

(b) put $x=-1$, $y=-4-k$

$5(-1)-2(-4-k)=11$

$-5+8+2k=11$

$k=4$

12. The equation of the parabola Γ is $y = x^2 + kx + k$, where k is a constant.
    Denote the straight line $y = -4x + 3$ by L.
    (a) Prove that L and Γ intersect at two distinct points. Prove $\Delta > 0$.
    (b) Suppose the points of intersection of L and Γ are A and B. Find the x-coordinate of the mid-point of AB in terms of k.

(a)

$x^2 + kx + k = -4x + 3$

$x^2 + (k+4)x + k - 3 = 0$

$\Delta = (k+4)^2 - 4 \cdot 1 \cdot (k-3)$

$= k^2 + 8k + 16 - 4k + 12$

$= k^2 + 4k + 28$

$= k^2 + 4k + 4 + 24$

$= (k+2)^2 + 24$

≥ 24

L and Γ intersect at two distinct points

(b) x-coord of mid-point of AB

$= \dfrac{-(k+4)}{1} \div 2$

$= \dfrac{-k-4}{2}$

13. The equation of the parabola Γ is $y = x^2 + kx + 2k$, where k is a constant. Denote the straight line $y = 4x + 12$ by L.

(a) Prove that L and Γ must have at least one point of intersection $\Delta \geq 0$

(b) Suppose L and Γ intersect at two distinct points A and B. If the mid-point of AB lies on the straight line $x = -1$, find k.

(a) $x^2 + kx + 2k = 4x + 12$

$x^2 + (k-4)x + 2k - 12 = 0$

$\Delta = (k-4)^2 - 4 \cdot 1 \cdot (2k-12)$

$= k^2 - 8k + 16 - 8k + 48$

$= k^2 - 16k + 64$

$= (k-8)^2, \geq 0$

$\therefore$ L and T must have at least one point of intersection

(b) x-coord of mid-point of AB

$= \dfrac{-(k-4)}{1} \div 2$

$= \dfrac{-k+4}{2}$

$\therefore \dfrac{-k+4}{2} = -1$

$-k + 4 = -2$

$k = 6$

Application Problems

14. A two-digit positive integer is increased by 36 when its digits are reversed. The square of the tens digit is 2 greater than the units digit. Find the original integer.

Let x be the tens digit and y be the units digit.

$x^2 - 2 = y \quad \cdots (1)$

$10x + y + 36 = 10y + x$

$9x - 9y + 36 = 0$

$x - y + 4 = 0 \quad \cdots (2)$

put (2) into (1)

$x - (x^2 - 2) + 4 = 0$

$-x^2 + x + 6 = 0$

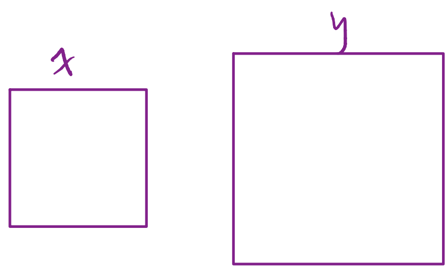
$x^2 - x - 6 = 0$

$x = -2$ (rej.) or 3

$\therefore \ y = 3^2 - 2 = 7$

$\therefore$ original number $= 37$

15. A wire of length 40 cm is cut into 2 pieces which are then bent to form 2 squares of sides x cm and y cm respectively, where $x < y$. If the total area of the two squares is 58 cm^2, find x and y.



$4x + 4y = 40$

$x + y = 10$

$x^2 + y^2 = 58$

$x^2 + (10 - x)^2 = 58$

$x^2 + 100 - 20x + x^2 = 58$

$2x^2 - 20x + 42 = 0$

$x^2 - 10x + 21 = 0$

$x = 3, \ y = 7$

9.3 Solving Equations that Can be Transformed into Quadratic Equations

A. Fractional Equations

Equations involving **algebraic fractions** such as $\frac{1}{x}+\frac{1}{x-1}=1$, etc. are called fractional equations. We can transform them into quadratic equations by multiplying both sides of the equation by the L.C.M. of all the denominators.

Steps	Example: Solve $\frac{4}{x}-\frac{1}{x-1}=1$.
1. Convert the fractions to fractions with the same denominator.	$\frac{4(x-1)}{x(x-1)}-\frac{x}{x(x-1)}=1$ $\frac{4x-4-x}{x(x-1)}=1$
2. Multiply both sides of the equation by a suitable algebraic expression to get rid of the denominator	$4x-4-x=x(x-1)$ $4x-4-x=x^2-x$ $x^2-4x+4=0$
3. We can get a quadratic equation.	$x^2-4x+4=0$ $(x-2)(x-2)=0$ $x=2$ (repeated)
If the solution makes some denominators in the original equation zero, it should be rejected.	

Example

16. Solve the following equations.

(a) $\frac{3}{x}-\frac{4}{x+1}=1$

$x(x+1)\left(\frac{3}{x}-\frac{4}{x+1}\right)=x(x+1)$

$3(x+1)-4x=x^2+x$

$3-x=x^2+x$

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3$ or 1

(b) $\frac{1}{x-3}-\frac{3}{x+2}=\frac{1}{2}$

$\frac{x+2-3(x-3)}{(x-3)(x+2)}=\frac{1}{2}$

$\frac{-2x+11}{x^2-x-6}=\frac{1}{2}$

$-4x+22=x^2-x-6$

$x^2+3x-28=0$

$(x+7)(x-4)=0$

$x=-7$ or 4

$x^2 - 9 = (x+3)(x-3)$

(c) $\dfrac{12}{x^2-9} - \dfrac{2}{x-3} = 1$

$x \neq 3, -3$

$\dfrac{12 - 2(x+3)}{x^2-9} = 1$

$12 - 2x - 6 = x^2 - 9$

$x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

$x = -5$ or 3 (rej.)

(d) $\dfrac{2}{x+3} + \dfrac{3}{x-3} = -1$

$\dfrac{2(x-3) + 3(x+3)}{(x+3)(x-3)} = -1$

$\dfrac{2x - 6 + 3x + 9}{x^2 - 9} = -1$

$5x + 3 = -x^2 + 9$

$x^2 + 5x - 6 = 0$

$x = -6$ or 1

17. Solve $\dfrac{1}{x^2-x} + \dfrac{1}{4(1-x)} = \dfrac{1}{4}$.

$1 - x = -(x-1)$

$x(x-1)$

$\dfrac{1}{x(x-1)} - \dfrac{1}{4(x-1)} = \dfrac{1}{4}$

$\dfrac{4 - x}{4x(x-1)} = \dfrac{1}{4}$

$16 - 4x = 4x^2 - 4x$

$16 = 4x^2$

$4 = x^2$

$x = \pm 2$

B. Equations of Higher Degree (with specific patterns)

By suitable substitution, some equations of degrees higher than 2 can be transformed into quadratic equations.

Example

18. Solve $x^4+2x^2-3=0$. $x^4=(x^2)^2$

Sol: Let $u=x^2$. The original equation can be written as

$u^2+2u-3=0$

$(u+3)(u-1)=0$

$u=-3$ or $u=1$

$x^2=-3$ (rej.) or $x^2=1$

$x=\pm 1$

Quadratic Equation in x^2

↓

$(x^2)^2+2x^2-3=0$

$(x^2+3)(x^2-1)=0$

$x^2=-3$ (rej.) or $x^2=1$

$x=\pm 1$

19. Solve the following equations.

(a) $x^4-5x^2+4=0$

$(x^2)^2-5x^2+4=0$

$(x^2-4)(x^2-1)=0$

$x^2=4$ or $x^2=1$

$x=\pm 2, \pm 1$

(b) $x^6+63x^3-64=0$

$(x^3)^2+63x^3-64=0$

$(x^3+64)(x^3-1)=0$

$x^3=-64$ or $x^3=1$

$x=-4$ or 1

(c) $4x^4+3x^2-1=0$

$4(x^2)^2+3x^2-1=0$

$(4x^2-1)(x^2+1)=0$

$x^2=\frac{1}{4}$ or $x^2=-1$ (rej.)

$x=\pm\frac{1}{2}$

(d) $8x^6-7x^3-1=0$

$8(x^3)^2-7x^3-1=0$

$(8x^3+1)(x^3-1)=0$

$x^3=-\frac{1}{8}$ or $x^3=1$

$x=-\frac{1}{2}$ or $x=1$

20. Solve the following equations. (*Leave your answers in surd form if necessary.*)

* (a) $x^7+63x^4-64x=0$

$x^6+63x^3-64=0$ X

$x(x^6+63x^3-64)=0$

$x[(x^3)^2+63x^3-64]=0$

$x(x^3-1)(x^3+64)=0$

$x=0$ or $x^3=1$ or $x^3=-64$

(b) $x^5+2x^3-8x=0$

$x(x^4+2x^2-8)=0$

$x[(x^2)^2+2x^2-8]=0$

$x(x^2+4)(x^2-2)=0$

$x=0$ or $x^2=-4$ (rej.) or $x^2=2$

$x=0$ or $x=\pm\sqrt{2}$

(c) $(x^2-5x)^2+3(x^2-5x)-54=0$

$(x^2-5x+9)(x^2-5x+6)=0$

$x^2-5x+9=0$ or $x^2-5x+6=0$

$\Delta=(-5)^2-4\cdot 9$; $(x+1)(x-6)=0$

$=-11<0$; $x=-1$ or 6

no real solution

Let $u=x^2-5x$

$u^2+3u-54=0$

$(u+9)(u-6)=0$

$u=-9$ or $u=6$

$x^2-5x=-9$ or $x^2-5x=6$

$x^2-5x+9=0$ or $x^2-5x-6=0$

$\Delta=(-5)^2-4\cdot 9$; $(x+1)(x-6)=0$

$=-11$; $x=-1$ or 6

no real solution

C. Equations with Square Root Signs

$\sqrt{4} = +2, \quad -\sqrt{4} = -2$

In solving an equation with a square root sign, we can transpose the term with the square root sign to one side of the equation, and then square both sides of the equation.

As **unwanted roots** may be created in this process, we need to check whether the roots obtained satisfy the origin equation.

$8 \cdot \frac{1}{4} + 2\sqrt{\frac{1}{4}} - 1 = 2 + 2\left(\frac{1}{2}\right) - 1 = 2, \neq 0$

$x = (\sqrt{x})^2$

Example

21. Solve the following equations.

Quadratic equation in $\sqrt{x}$

(a) $x - 6\sqrt{x} + 8 = 0$ ← Substitution → (b) $8x + 2\sqrt{x} - 1 = 0$

(a)

Let $u = \sqrt{x}$

$u^2 - 6u + 8 = 0$

$(u-2)(u-4) = 0$

$u = 2$ or 4

$\sqrt{x} = 2$ or $\sqrt{x} = 4$

$x = 2^2 = 4$ or $x = 4^2 = 16$

(b)

$8(\sqrt{x})^2 + 2\sqrt{x} - 1 = 0$

$(4\sqrt{x} - 1)(2\sqrt{x} + 1) = 0$

$\sqrt{x} = \frac{1}{4}$ or $\sqrt{x} = -\frac{1}{2}$ (rej.)

$x = \frac{1}{16}$

$x = \frac{1}{4}$

$\sqrt{\ }$ ← positive square root.

(c) $x - \sqrt{2x-1} = 8$ ← remove $\sqrt{\ }$ directly → (d) $10 - 2\sqrt{x-2} = x$

(c)

$(x - 2\sqrt{2x-1})^2 = 8^2$ ✗

$x^2 - 4x\sqrt{2x-1} + 4(2x-1) = 64$

$x - 8 = \sqrt{2x-1}$

$(x-8)^2 = 2x-1$

$x^2 - 16x + 64 - 2x + 1 = 0$

$x^2 - 18x + 65 = 0$

$(x-13)(x-5) = 0$

$x = 13$ or 5 (rej.)

put $x = 5$

L.H.S. $= 5 - \sqrt{2 \cdot 5 - 1}$

$= 5 - \sqrt{9}$

$= 5 - 3 = 2, \neq 8$

(d)

$10 - x = 2\sqrt{x-2}$

$(10-x)^2 = (2\sqrt{x-2})^2$

$100 - 20x + x^2 = 4(x-2)$

$100 - 20x + x^2 - 4x + 8 = 0$

$x^2 - 24x + 108 = 0$

$(x-6)(x-18) = 0$

$x = 6$ or 18 (rej.)

$$x - 6\sqrt{x} + 8 = 0$$
$$x + 8 = 6\sqrt{x}$$
$$(x+8)^2 = (6\sqrt{x})^2$$
$$x^2 + 16x + 64 = 36x$$
$$x^2 - 20x + 64 = 0$$
$$(x-4)(x-16) = 0$$
$$x = 4 \quad \text{or} \quad 16$$

$$8x + 2\sqrt{x} - 1 = 0$$
$$2\sqrt{x} = 1 - 8x$$
$$(2\sqrt{x})^2 = (1-8x)^2$$
$$4x = 1 - 16x + 64x^2$$
$$64x^2 - 20x + 1 = 0$$
$$(16x-1)(4x-1) = 0$$
$$x = \frac{1}{16} \quad \text{or} \quad x = \frac{1}{4} \text{ (rej.)}$$

D. Exponential Equations (with specific pattern)

By using the laws of indices and suitable substitution, some exponential equations can be reduced to quadratic equations. $a^{2x} = (a^2)^x$, a^x $a^x > 0$

Example

22. Solve the following equations.
(*Give the answer correct to 3 significant figures if necessary.*)

(a) $2(2^{2x}) - 3(2^x) + 1 = 0$

$2(2^x)^2 - 3(2^x) + 1 = 0$

Let $u = 2^x$

$2u^2 - 3u + 1 = 0$

$(2u-1)(u-1) = 0$

$u = \frac{1}{2}$ or $u = 1$

$2^x = \frac{1}{2}$ or $2^x = 1$

$x = -1$ or 0

(b) $5^{2x} - 30(5^x) + 125 = 0$

Quadratic equation in 5^x

$(5^x)^2 - 30 \cdot 5^x + 125 = 0$

$(5^x - 25)(5^x - 5) = 0$

$5^x = 25$ or $5^x = 5$

$x = 2$ or 1

$4^x = (2^2)^x = (2^x)^2$

(c) $4^x - 2^x - 56 = 0$

$(2^x)^2 - 2^x - 56 = 0$

$(2^x + 7)(2^x - 8) = 0$

$2^x = -7$ (rej.) or $2^x = 8$

$(1.21)^x = (1.1^2)^x = (1.1^x)^2$

(d) $1.21^x - 10(1.1^x) + 24 = 0$

$(1.1^x)^2 - 10(1.1^x) + 24 = 0$

$(1.1^x - 4)(1.1^x - 6) = 0$

$1.1^x = 4$ or $1.1^x = 6$

$x \log 1.1 = \log 4$ or $x \log 1.1 = \log 6$

$x = 14.5$ or 18.8

$(3^x)^2 = 3^{2x} = 9^x \quad 3^x$

(e) $2(2^x) - 5 - \frac{3}{2^x} = 0$

$2(2^x)^2 - 5(2^x) - 3 = 0$

$[2(2^x) + 1](2^x - 3) = 0$

$2^x = -\frac{1}{2}$ (rej.) or $2^x = 3$

$x = \frac{\log 3}{\log 2}$

$x = 1.58$

(f) $9^{x+1} + 7(3^x) - 2 = 0$

$9 \cdot 9^x + 7(3^x) - 2 = 0$

$9 \cdot (3^x)^2 + 7(3^x) - 2 = 0$

$[9 \cdot (3^x) - 2](3^x + 1) = 0$

$3^x = \frac{2}{9}$ or $3^x = -1$ (rej.)

$x \log 3 = \log \frac{2}{9}$

$x = \log \frac{2}{9} \div \log 3$

$x = -1.37$

E. Logarithmic Equations

By using the properties of logarithms and suitable substitution, some logarithmic equations can be reduced to quadratic equations.

Example

23. Solve the following equations.

quadratic equation in $\log x$.

(a) $(\log x)^2 - 6\log x + 8 = 0$

$(\log x - 2)(\log x - 4) = 0$

$\log x = 2$ or $\log x = 4$

$x = 10^2$ or $x = 10^4$

$x = 100$ or 10000

(b) $(\log_3 x)^2 - \log_3 x^4 + 4 = 0$

$(\log_3 x)^2 - 4\log_3 x + 4 = 0$

$(\log_3 x - 2)^2 = 0$

$\log_3 x = 2$

$x = 3^2 = 9$

24. Solve the following expressions.

(a) $\log(2x-1)+\log(x-1)=1$

$\log[(2x-1)(x-1)]=1$

$2x^2-2x-x+1=10$

$2x^2-3x-9=0$

$(x-3)(2x+3)=0$

$x=3$ or $x=-\frac{3}{2}$ (rej.)

(b) $\log_2 x-\log_2(x^2-21)=-2$

$\log_2 \frac{x}{x^2-21}=-2$

$\frac{x}{x^2-21}=2^{-2}$

$\frac{x}{x^2-21}=\frac{1}{4}$

$4x=x^2-21$

$x^2-4x-21=0$

$(x+3)(x-7)=0$

$x=-3$ (rej.) or $x=7$

25. Solve $\frac{1}{\log x+3}+\frac{4}{\log x-3}=-1$.

$\frac{\log x-3+4(\log x+3)}{(\log x+3)(\log x-3)}=-1$

$\frac{\log x-3+4\log x+12}{(\log x)^2-9}=-1$

$5\log x+9=-(\log x)^2+9$

$(\log x)^2+5\log x=0$

$\log x(\log x+5)=0$

$\log x=0$ or $\log x=-5$

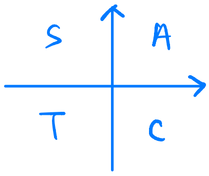
$x=10^0$ or 10^{-5}

$x=1$ or $\frac{1}{100000}$

F. Trigonometric Equations

$-1 \le \sin x \le 1$, $-1 \le \cos x \le 1$

Some trigonometric equations can be **reduced to quadratic equations** first and then be solved.



Example

26. Solve the following equations, where $0° \le x \le 360°$.

(a) $2\sin^2 x - 3\sin x - 2 = 0$

$(2\sin x + 1)(\sin x - 2) = 0$

$\sin x = -\frac{1}{2}$ or $\sin x = 2$ (rej.)

$x = 180° + \sin^{-1}\frac{1}{2}$, $360° - \sin^{-1}\frac{1}{2}$

$x = 210°$, $330°$

(b) $4\cos^2 x - 4\cos x - 3 = 0$

$(2\cos x + 1)(2\cos x - 3) = 0$

$\cos x = -\frac{1}{2}$ or $\cos x = \frac{3}{2}$ (rej.)

$x = 180° - \cos^{-1}\frac{1}{2}$, $180° + \cos^{-1}\frac{1}{2}$

$x = 120°$, $240°$

(c) $2\sin^2 x - 7\sin x - 4 = 0$

$(2\sin x + 1)(\sin x - 4) = 0$

$\sin x = -\frac{1}{2}$ or $\sin x = 4$ (rej.)

$x = 180° + \sin^{-1}\frac{1}{2}$, $360° - \sin^{-1}\frac{1}{2}$

$= 210°$, $330°$

(d) $\tan^2 x = \tan x$

$\tan x = 1$ ✗

$\tan^2 x - \tan x = 0$

$\tan x(\tan x - 1) = 0$

$\tan x = 0$ or $\tan x = 1$

$x = 0°, 180°, 360°, \tan^{-1}1, 180° + \tan^{-1}1$

$x = 0°, 45°, 180°, 225°, 360°$

*	$\sin x = 0$,	$x = 0°, 180°, 360°$
	$\cos x = 1$,	$x = 0°, 360°$
	$\tan x = 0$,	$x = 0°, 180°, 360°$

27. Solve the following equations, where $0° \le x < 360°$. ← exclude $360°$

(*Give the answers correct to 1 decimal place if necessary*.)

(a) $2\cos^2 x = 3 - 3\sin x$

$2(1-\sin^2 x) = 3 - 3\sin x$

$2 - 2\sin^2 x = 3 - 3\sin x$

$2\sin^2 x - 3\sin x + 1 = 0$

$(2\sin x - 1)(\sin x - 1) = 0$

$\sin x = \frac{1}{2}$ or $\sin x = 1$

$x = \sin^{-1}\frac{1}{2},\ 180° - \sin^{-1}\frac{1}{2},\ 90°$

$= 30°, 90°, 150°$

$\left.\begin{array}{l}\sin x \to \cos x \\ \cos x \to \sin x\end{array}\right\}$ ✗

$\left.\begin{array}{l}\sin^2 x \to \cos^2 x \\ \cos^2 x \to \sin^2 x\end{array}\right\}$ ✓

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

(b) $2\sin^2 x - \cos x = 1$

$2(1-\cos^2 x) - \cos x = 1$

$2 - 2\cos^2 x - \cos x - 1 = 0$

$2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

$x = \cos^{-1}\frac{1}{2},\ 360° - \cos^{-1}\frac{1}{2},\ 180°$

$= 60°, 180°, 300°$

(c) $2\sin^2 x + \cos x = \cos^2 x$

$2 - 2\cos^2 x + \cos x = \cos^2 x$

$3\cos^2 x - \cos x - 2 = 0$

$(3\cos x + 2)(\cos x - 1) = 0$

$\cos x = -\frac{2}{3}$ or $\cos x = 1$

$x = 180° - \cos^{-1}\frac{2}{3},\ 180° + \cos^{-1}\frac{2}{3},\ 0°,\ \cancel{360°}$

$= 0°,\ 131.8°,\ 228.2°$

(d) $3\sin^2 x + 2\sin x\cos x - \cos^2 x = 0$

$3a^2 + 2ab - b^2 = 0$

$(3a - b)(a + b) = 0$

$(3\sin x - \cos x)(\sin x + \cos x) = 0$

$3\sin x - \cos x = 0$ or $\sin x + \cos x = 0$

$3\sin x = \cos x$ $\quad$ $\sin x = -\cos x$

$\tan x = \frac{1}{3}$ $\quad$ $\tan x = -1$

$x = \tan^{-1}\frac{1}{3},\ 180° + \tan^{-1}\frac{1}{3},\ 180° - \tan^{-1}1,\ 360° - \tan^{-1}1$

$x = 18.4°,\ 198.4°,\ 135°,\ 315°$

9.4 Solving Problems Leading to Quadratic Equations

Example

28. A group of graduates plans to spend \$4800 for the re-union party. When 2 more graduates confirm to join the party, each graduate in the original group can pay \$10 less. Find the number of graduates in the original group.

Let x be the original no. of graduates.

$$\frac{4800}{x} - \frac{4800}{x+2} = 10$$

$$4800(x+2) - 4800x = 10x(x+2)$$

$$4800x + 9600 - 4800x = 10x^2 + 20x$$

$$0 = x^2 + 2x - 960$$

$$0 = (x+32)(x-30)$$

$$x = -32 \text{ (rej.) or } 30$$

29. Jack cycles from place A to place B at a constant speed. If he increases his speed by 3 km/h, he will arrive at B half an hour earlier. Given that the distance between A and B is 30 km, find Jack's original cycling speed.

Let x km/h be the original cycling speed.

$$\frac{30}{x} - \frac{30}{x+3} = \frac{1}{2}$$

$$\frac{30(x+3) - 30x}{x(x+3)} = \frac{1}{2}$$

$$90 \times 2 = x^2 + 3x$$

$$0 = x^2 + 3x - 180$$

$$0 = (x+15)(x-12)$$

$$x = -15 \text{ (rej.) or } 12$$

$0.81 = 0.9^2$ $\quad (1-10\%) = 0.9 \quad (1-19\%) = 0.81$

30. Watch A and Watch B are worth \$12000 and \$16000 at present respectively. The depreciation rates of watch A and watch B are 10% and 19% respectively. After how many years will the total value of the two watches be \$10800? (*Round up the answer correct to the nearest integer*.)

Let n be the no. of years

$$12000 \times 0.9^n + 16000 \times 0.81^n = 10800$$

$$30 \times 0.9^n + 40 \times (0.9)^{2n} = 27$$

$$40 \times (0.9^n)^2 + 30 \times 0.9^n - 27 = 0$$

$$0.9^n = 0.52819593$$

$$n = \frac{\log 0.52819593}{\log 0.9}$$

$$n = 7$$