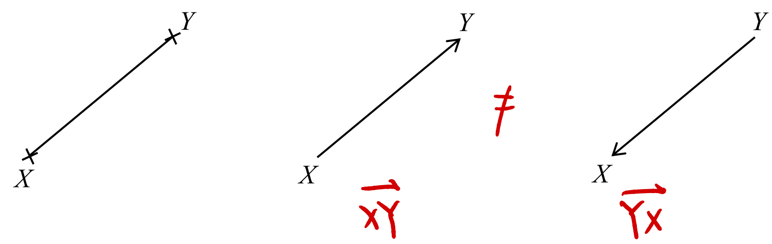
Chapter 12 Introduction to Vectors
Supplementary Notes

Name: ______________________________ (   )   Class: F.5 _____

12.1 Concepts of Vectors

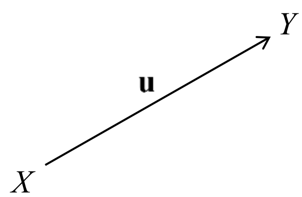
A. Scalars and Vectors

- Some quantities can be described by magnitudes (numerical values) alone, such as temperature, length and time. These quantities are called __scalars__.
- Some quantities have to be described by both magnitude and __direction__, such as displacement, velocity and force. These quantities are called __vector__.



B. Representation of Vectors

A vector is represented by a directed line segment.



The figure shows a vector from an __Initial__ point X to a __terminal__ point Y.

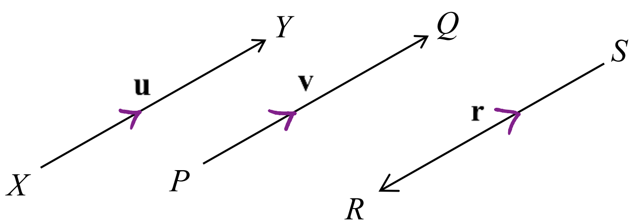
This vector can be denoted by $\overrightarrow{XY}$, $\mathbf{u}$ or $\vec{u}$. $\overrightarrow{XY}$, $\vec{u}$

Its magnitude can be denoted by $|\overrightarrow{XY}|$, $|\mathbf{u}|$ or $|\vec{u}|$. $|\overrightarrow{XY}|$

C. Equality of Vectors and Negative Vector

Two vectors are equal if and only if they have the same __magnitude__ and the same __direction__.

The **negative vector** of $\mathbf{u}$ has the same magnitude but in the opposite direction of $\mathbf{u}$. It is denoted by $-\mathbf{u}$.



$\mathbf{v} =$ __$\vec{u}$__ $\overrightarrow{PQ} =$ __$\overrightarrow{XY}$__

$\mathbf{r} =$ __$-\vec{u}$__ $\overrightarrow{SR} =$ __$-\overrightarrow{PQ}$__

Example

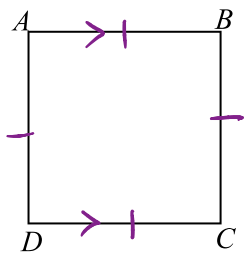
1 The figure shows a square *ABCD*. Determine whether the following sentences are correct.

(a) $\overrightarrow{AB}=\overrightarrow{CD}$ × $\overrightarrow{AB}=\overrightarrow{DC}$, $\overrightarrow{AB}=-\overrightarrow{CD}$

(b) $\overrightarrow{AB}=\overrightarrow{AD}$ ×

(c) $\left|\overrightarrow{AB}\right|=\left|\overrightarrow{CD}\right|$ ✓

length of AB, CD



D. Zero Vector and Unit Vector

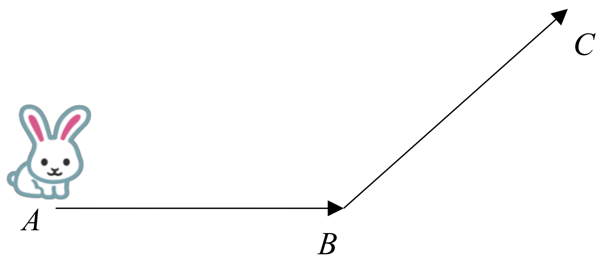
$\vec{0}$

A vector with ___0___ magnitude is called a **zero vector** is denoted by $\mathbf{0}$. Zero vectors do not have any specified direction.

A vector with magnitude ___1___ is called a **unit vector**, i.e. $\mathbf{u}$ is a unit vector when $|\mathbf{u}|=$ ___1___ and is denoted by $\hat{\mathbf{u}}$. $\hat{u}$

12.2 Basic Operations of Vectors

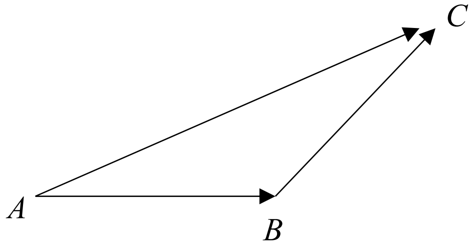
A. Addition of Vectors



Definition:

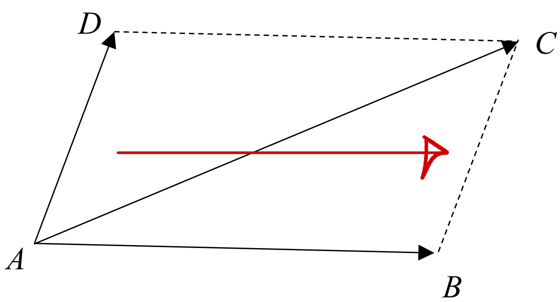
$\overrightarrow{AB}=\overrightarrow{DC}$, $\overrightarrow{AD}=\overrightarrow{BC}$

Triangle Law of Addition



$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

Parallelogram Law of Addition

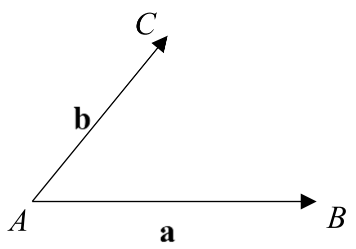


$\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

B. Subtraction of Vectors

+/- ⇒ direction

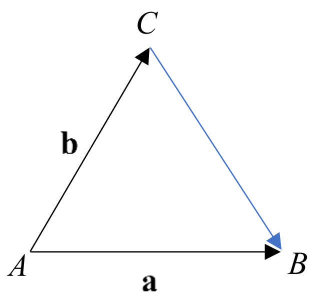


$\vec{AB} = \vec{a}$

$\vec{AC} = \vec{b}$

$\vec{a} - \vec{b}$
$= \vec{AB} - \vec{AC}$
$= \vec{AB} + (-\vec{AC})$
$= \vec{AB} + \vec{CA}$
$= \vec{CA} + \vec{AB}$
$= \vec{CB}$

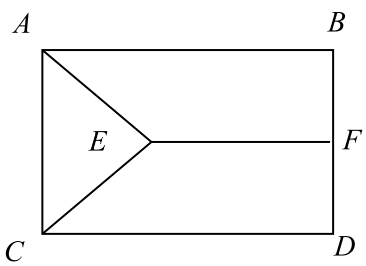
Subtraction of Vectors



$$\overrightarrow{CB} = \mathbf{a} - \mathbf{b}$$

<u>Example</u>

2. In the figure, *ABCD* is a rectangle. Express each of the following as a single vector.



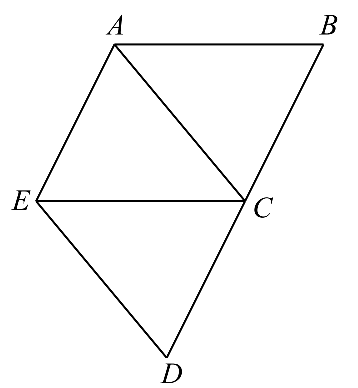
(a) $\overrightarrow{AB} + \overrightarrow{BF} + \overrightarrow{FD}$

$= \vec{AF} + \vec{FD}$
$= \vec{AD}$

(b) $\overrightarrow{AE} - \overrightarrow{CE} - \overrightarrow{BC}$

$= \vec{AE} + \vec{EC} + \vec{CB}$
$= \vec{AC} + \vec{CB} = \vec{AB}$

3. In the figure, *ABCE* is a parallelogram. Express each of the following as a single vector.



(a) $\overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{CD}$

$= \vec{AB} + \vec{BC} + \vec{CD}$
$= \vec{AC} + \vec{CD} = \vec{AD}$

(b) $\overrightarrow{CE} - \overrightarrow{CA} + \overrightarrow{AB}$

$= \vec{CE} + \vec{AC} + \vec{AB}$
$= \vec{AC} + \vec{CE} + \vec{AB}$
$= \vec{AE} + \vec{AB}$
$= \vec{AE} + \vec{EC}$
$= \vec{AC}$

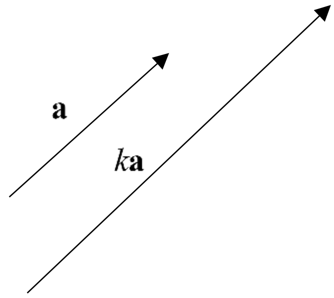
C. Scalar Multiplication of Vectors and Parallelism

Definition:

It is given that $\mathbf{a}$ is a non-zero vector.

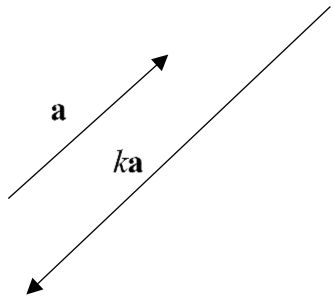
(i) If $k > 0$, then the direction of $k\mathbf{a}$ is the same as that of $\mathbf{a}$, and the magnitude of $k\mathbf{a}$ is k times the magnitude of $\mathbf{a}$.

i.e. $|k\mathbf{a}| = k|\mathbf{a}|$.



(ii) If $k < 0$, then the direction of $k\mathbf{a}$ is opposite to that of $\mathbf{a}$, and the magnitude of $k\mathbf{a}$ is $-k$ times the magnitude of $\mathbf{a}$.

i.e. $|k\mathbf{a}| = -k|\mathbf{a}|$



(iii) If $k = 0$, then $k\mathbf{a} = \mathbf{0}$.

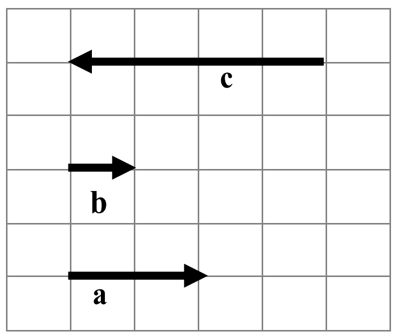
* (iv) $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ is a **unit vector** which has the same direction as $\mathbf{a}$.

<u>Example</u>

4. In the figure, express $\mathbf{b}$ and $\mathbf{c}$ in terms of $\mathbf{a}$.

$\vec{b} = \frac{1}{2}\vec{a}$

$\vec{c} = -2\vec{a}$



Properties of and scalar multiplication of vectors:

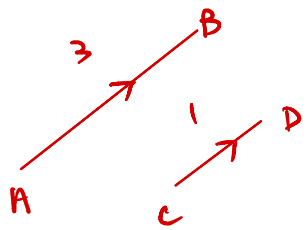
For any vectors $\mathbf{a}$, $\mathbf{b}$ and scalars λ, μ,

(a) $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a} = \mu(\lambda\mathbf{a})$

(b) $(\lambda \pm \mu)\mathbf{a} = \lambda\mathbf{a} \pm \mu\mathbf{a}$

(c) $\lambda(\mathbf{a} \pm \mathbf{b}) = \lambda\mathbf{a} \pm \lambda\mathbf{b}$

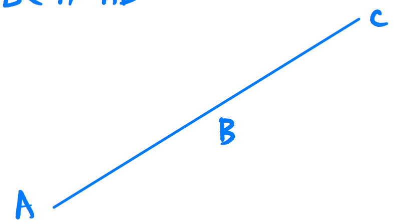
Two non-zero vectors **a and b** are parallel if and only if $\mathbf{b} = \lambda\mathbf{a}$ for some non-zero scalars λ.
In particular, for three distinct points *A*, *B* and *C*, if $\overrightarrow{BC} = \lambda\overrightarrow{AB}$, then *A*, *B* and *C* are collinear.



$\overrightarrow{AB} = 3\overrightarrow{CD}$

$\frac{1}{3}\overrightarrow{AB} = \overrightarrow{CD}$

$BC \parallel AB$

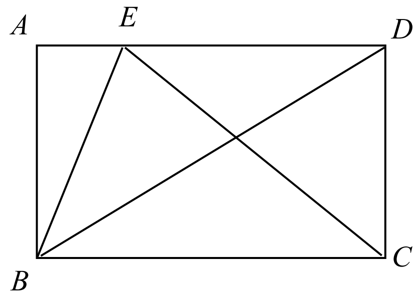


$\overrightarrow{DE} = 4\overrightarrow{AE}$

Example

5. In the figure, *ABCD* is a rectangle. *E* is a point on *AD* such that *DE* = 4*AE*. Let $\mathbf{CD} = \mathbf{u}$ and $\mathbf{BD} = \mathbf{v}$. Express the following vectors in terms of $\mathbf{u}$ and $\mathbf{v}$.

1 : 4



(a) **BC**

$= \overrightarrow{BD} + \overrightarrow{DC}$

$= \vec{v} + (-\vec{u})$

$= \vec{v} - \vec{u}$

(b) **DE**

$\overrightarrow{BC} = \overrightarrow{AD} = \vec{v} - \vec{u}$

$\overrightarrow{DA} = \vec{u} - \vec{v}$

$\overrightarrow{DE} = \frac{4}{5}\overrightarrow{DA} = \frac{4}{5}(\vec{u} - \vec{v})$

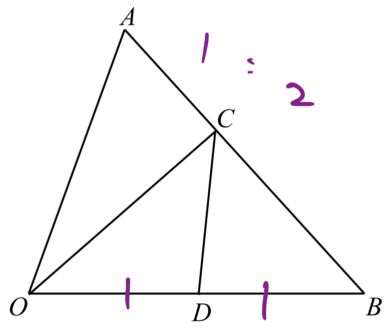
(c) **BE**

$= \overrightarrow{BD} + \overrightarrow{DE}$

$= \vec{v} + \frac{4}{5}\vec{u} - \frac{4}{5}\vec{v}$

$= \frac{4}{5}\vec{u} + \frac{1}{5}\vec{v}$

6. In the figure, D is the mid-point of OB. C is a point on AB such that $AC : CB = 1 : 2$. Let $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$, express $\mathbf{OC}$ and $\mathbf{CD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



$\vec{AB} = \vec{AO} + \vec{OB} = -\vec{a} + \vec{b}$

$\vec{AC} = \frac{1}{3}\vec{AB} = \frac{1}{3}\vec{b} - \frac{1}{3}\vec{a}$

$\vec{OC} = \vec{OA} + \vec{AC}$

$= \vec{a} + \frac{1}{3}\vec{b} - \frac{1}{3}\vec{a}$

$= \frac{2}{3}\vec{a} + \frac{1}{3}\vec{b}$

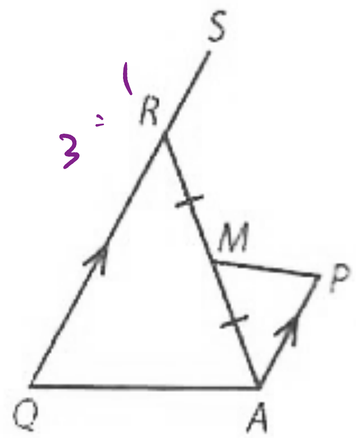
$\vec{OD} = \frac{1}{2}\vec{OB} = \frac{1}{2}\vec{b}$

$\vec{CD} = \vec{CO} + \vec{OD} = -\frac{2}{3}\vec{a} - \frac{1}{3}\vec{b} + \frac{1}{2}\vec{b} = \frac{1}{6}\vec{b} - \frac{2}{3}\vec{a}$

7. In the figure, R is a point on QS such that $QR : RS = 3 : 1$ and M is the mid-point of AR. $AP \parallel QS$ and $QS = 3AP$. It is given that $\overrightarrow{AP} = \mathbf{p}$ and $\overrightarrow{AQ} = \mathbf{q}$.

(a) Express $\overrightarrow{AR}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

(b) Express $\overrightarrow{MP}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.



(a) $\vec{QS} = 3\vec{AP} = 3\vec{p}$

$\vec{QR} = \frac{3}{4}\vec{QS} = \frac{9}{4}\vec{p}$

$\vec{AR} = \vec{AQ} + \vec{QR} = \vec{q} + \frac{9}{4}\vec{p}$

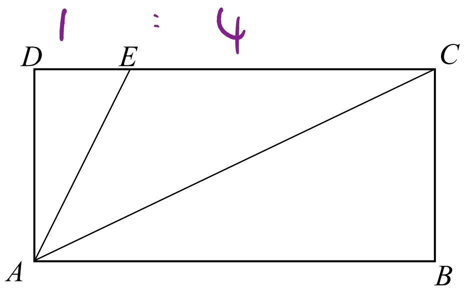
(b) $\vec{AM} = \frac{1}{2}\vec{AR} = \frac{1}{2}\vec{q} + \frac{9}{8}\vec{p}$

$\vec{MP} = \vec{MA} + \vec{AP} = -\vec{AM} + \vec{AP}$

$= -\frac{1}{2}\vec{q} - \frac{9}{8}\vec{p} + \vec{p}$

$= -\frac{1}{8}\vec{p} - \frac{1}{2}\vec{q}$

8. In the figure, $ABCD$ is a rectangle. E is a point on CD such that $CE = 4DE$. If $\overrightarrow{AD} = \mathbf{p}$ and $\overrightarrow{AB} = \mathbf{q}$, express $5\overrightarrow{AE} + 2\overrightarrow{CA}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.



$\overrightarrow{DC} = \overrightarrow{AB} = \vec{q}$

$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE} = \vec{p} + \frac{1}{5}\overrightarrow{DC} = \vec{p} + \frac{1}{5}\vec{q}$

$\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA} = -\overrightarrow{BC} - \overrightarrow{AB} = -\vec{p} - \vec{q}$

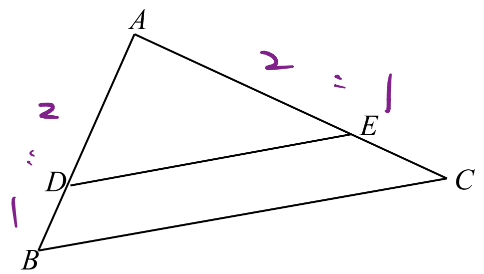
$\therefore 5\overrightarrow{AE} + 2\overrightarrow{CA} = 5\vec{p} + \vec{q} - 2\vec{p} - 2\vec{q}$

$= 3\vec{p} - \vec{q}$

9. In the figure, D and E are two points on AB and AC respectively such that $AD : DB = AE : EC = 2 : 1$. Let $\mathbf{AB} = \mathbf{b}$ and $\mathbf{AC} = \mathbf{c}$.

(a) Express $\mathbf{DE}$ in terms of $\mathbf{b}$ and $\mathbf{c}$.

(b) Hence prove that $DE \parallel BC$ and find $DE : BC$.



(a) $\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE}$

$= -\overrightarrow{AD} + \overrightarrow{AE}$

$= -\frac{2}{3}\overrightarrow{AB} + \frac{2}{3}\overrightarrow{AC}$

$= \frac{2}{3}(\vec{c} - \vec{b})$

(b) $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= -\vec{b} + \vec{c}$

$= \vec{c} - \vec{b}$

$= \frac{3}{2} \cdot \overrightarrow{DE}$

$\therefore BC \parallel DE$

$\frac{\overrightarrow{BC}}{\overrightarrow{DE}} = \frac{3}{2}$ ✗

$\overrightarrow{DE} = \frac{2}{3}\overrightarrow{BC}$

$|\overrightarrow{DE}| = \frac{2}{3}|\overrightarrow{BC}|$

$DE : BC = 2 : 3$

D. More About Properties of Vectors

Let $\mathbf{a}$, $\mathbf{b}$ ne two non-zero and non-parallel vectors, and λ, μ be scalars.

(a) If $\lambda\mathbf{a}+\mu\mathbf{b}=\mathbf{0}$, then $\lambda=\mu=0$;

(b) Suppose λ_1, λ_2, μ_1, μ_2 are non-zero scalars.

If $\lambda_1\mathbf{a}+\mu_1\mathbf{b}=\lambda_2\mathbf{a}+\mu_2\mathbf{b}$, then $\lambda_1=\lambda_2$ and $\mu_1=\mu_2$.

Example

10. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors which are not parallel to each other.
If $\mathbf{u}=2\mathbf{v}$, where $\mathbf{u}=(5k-1)\mathbf{a}+(k-3)\mathbf{b}$ and $\mathbf{v}=(5+3n)\mathbf{a}+n\mathbf{b}$, find the values of k and n.

$2\vec{v} = (10+6n)\vec{a} + 2n\vec{b}$

$\vec{u} = 2\vec{v}$, $\quad 10+6n = 5k-1 \quad 2n = k-3 \quad k=1,\ n=-1$

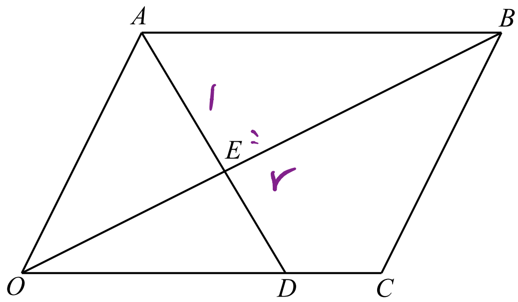
$5k-6n = 11 \quad k-2n=3$

11. In the figure, $OABC$ is a parallelogram with $\mathbf{OA}=\mathbf{a}$ and $\mathbf{OC}=\mathbf{c}$. D is a point on OC such that $OD:DC=2:1$.

(a) If $AE:ED=1:r$ ~~AE:ED=1:r~~, express $\mathbf{OE}$ in terms of r, $\mathbf{a}$ and $\mathbf{c}$.

(b) If $OE=sOB$, express $\mathbf{OE}$ in terms of s, $\mathbf{a}$ and $\mathbf{c}$.

(c) Find the values of r and s.



2 : 1

(a) $\vec{OD} = \frac{2}{3}\vec{OC} = \frac{2}{3}\vec{c}$

$\vec{AD} = \vec{AO}+\vec{OD} = -\vec{a}+\frac{2}{3}\vec{c}$

$\vec{AE} = \frac{1}{1+r}\vec{AD} = \frac{-1}{1+r}\vec{a}+\frac{2}{3(1+r)}\vec{c}$

$\vec{OE} = \vec{OA}+\vec{AE} = \vec{a}+\left(-\frac{1}{1+r}\vec{a}+\frac{2}{3(1+r)}\vec{c}\right)$

$= \frac{r}{1+r}\vec{a}+\frac{2}{3(1+r)}\vec{c}$

(b) $\vec{OB} = \vec{OA}+\vec{AB} = \vec{OA}+\vec{OC} = \vec{a}+\vec{c}$

$\vec{OE} = s\vec{OB} = s\vec{a}+s\vec{c}$

(c) $\frac{r}{1+r} = s$, $\quad \frac{2}{3(1+r)} = s$

$\frac{r}{1+r} = \frac{2}{3(1+r)}$

$r = \frac{2}{3}$

$s = \frac{\frac{2}{3}}{1+\frac{2}{3}} = \frac{2}{5}$

$\vec{u} = 2\vec{i} + 3\vec{j}$, $\vec{v} = 4\vec{i} + 6\vec{j}$

Recall: For non-zero vectors **u** and **v**, **u** is ***parallel*** to **v** if and only if $\boxed{\mathbf{u} = k\mathbf{v}}$, where $k \neq 0$.

➔ **It is given that** $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ **and** $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$. **u is parallel to v if**

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

12. It is given that the vectors $\overrightarrow{PQ} = 2\mathbf{a} + m(\mathbf{a}+\mathbf{b})$ and $\overrightarrow{RS} = -3\mathbf{a} + (m-2)\mathbf{b}$ are parallel, where **a** and **b** are two non-zero and non-parallel vectors and m is a scalar. Find the values of m.

$\overrightarrow{PQ} = 2\vec{a} + m\vec{a} + m\vec{b}$

$= (m+2)\vec{a} + m\vec{b}$

$\because \overrightarrow{PQ} \parallel \overrightarrow{RS}$, $\dfrac{m+2}{-3} = \dfrac{m}{m-2}$

$m^2 - 4 = -3m$

$m^2 + 3m - 4 = 0$

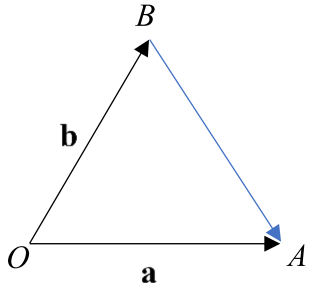
$(m+4)(m-1) = 0$

$m = -4$ or 1

12.3 More About Operations of Vectors

A. Position Vectors

It is given that A is a point on a plane π. If we take any point O on the plane as a reference point, then the position A can be represented by the vector $\overrightarrow{OA}$. Thus $\overrightarrow{OA}$ is called the **position vector** of A with respect to O.



$\overrightarrow{AO} + \overrightarrow{OB}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$$

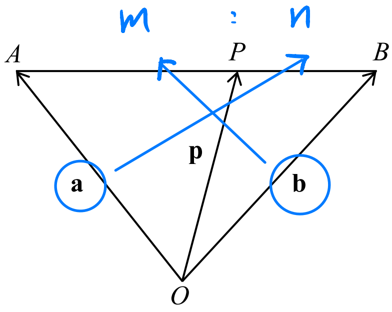
Example $\vec{OA}, \vec{OB}, \vec{OC}, \vec{OD}$

13. Let A, B, C and D be any four points. Using position vectors with the reference point O, show that $\overrightarrow{AC}+\overrightarrow{BD}=\overrightarrow{AD}+\overrightarrow{BC}$.

$\vec{AC}+\vec{BD} = \vec{AO}+\vec{OC}+\vec{BO}+\vec{OD} = \vec{AO}+\vec{OD}+\vec{BO}+\vec{OC} = \vec{AD}+\vec{DC}$

B. Division of a Line Segment

Section Formula for Vectors

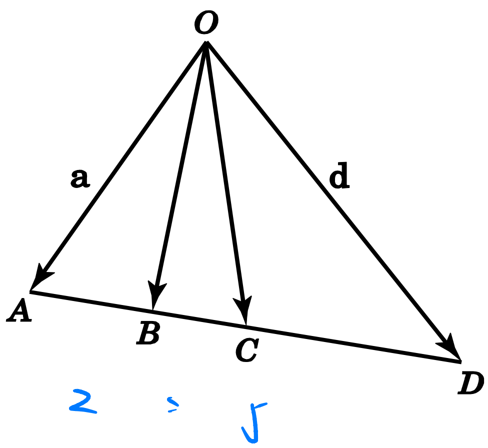


(i) If $AP:PB=m:n$, then $\mathbf{p}=\dfrac{n\mathbf{a}+m\mathbf{b}}{m+n}$.

(ii) If P is the mid-point of AB, then $\mathbf{p}=\dfrac{\mathbf{a}+\mathbf{b}}{2}$.

Example

14. In the figure, B is a point on AD such that AB : BD = 2 : 5 and C is the mid-point of AD. The position vectors of A and D with respect to O are **a** and **d** respectively. Express each of the following in terms of **a** and **d**.
   (a) **OB**
   (b) **OC**
   (c) **BC**



(a) $\vec{OB} = \dfrac{5\vec{a}+2\vec{d}}{5+2} = \dfrac{5}{7}\vec{a}+\dfrac{2}{7}\vec{d}$

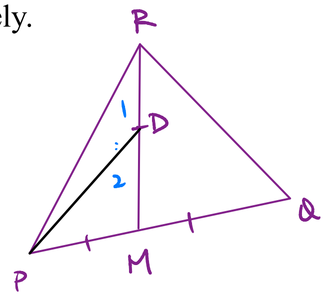
(b) $\vec{OC} = \dfrac{\vec{a}+\vec{d}}{2} = \dfrac{1}{2}\vec{a}+\dfrac{1}{2}\vec{d}$

(c). $\vec{BC} = \vec{BO}+\vec{OC} = -\vec{OB}+\vec{OC}$
$= -\dfrac{5}{7}\vec{a}-\dfrac{2}{7}\vec{d}+\dfrac{1}{2}\vec{a}+\dfrac{1}{2}\vec{d}$
$= \dfrac{-3}{14}\vec{a}+\dfrac{3}{14}\vec{d}$

MD : DR = 2 : 1

15. In ΔPQR, M is the mid-point of PQ. D is a point on MR such that $MD = 2DR$. The position vectors of P, Q, R, D and M with respect to O are **p**, **q**, **r**, **d** and **m** respectively.
    (a) Express **m** in terms of **p** and **q**.
    (b) Express **d** in terms of **p**, **q** and **r**.

$\vec{OP}, \vec{OQ}, \vec{OR}, \vec{OD}, \vec{OM}$



(a) $\vec{RM} = \dfrac{\vec{RP} + \vec{RQ}}{2}$

$\vec{RO} + \vec{OM} = \dfrac{\vec{RO} + \vec{OP} + \vec{RO} + \vec{OQ}}{2}$

$-\vec{r} + \vec{m} = \dfrac{-\vec{r} + \vec{p} - \vec{r} + \vec{q}}{2}$

$\cancel{-\vec{r}} + \vec{m} = \cancel{-\vec{r}} + \dfrac{\vec{p}+\vec{q}}{2}$

$\vec{m} = \dfrac{\vec{p}+\vec{q}}{2}$

(b) $\vec{PD} = \dfrac{2\vec{PR} + \vec{PM}}{3}$

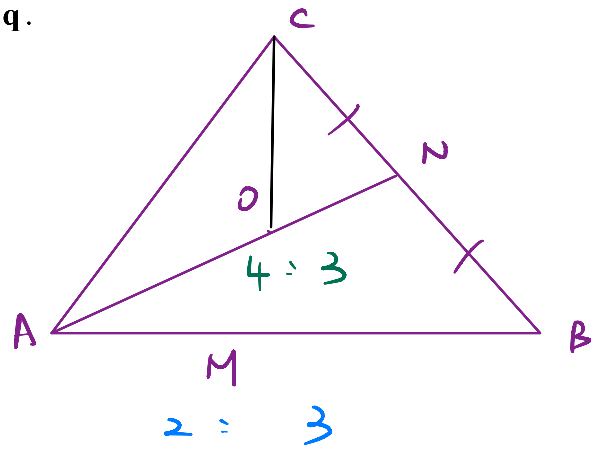
$\vec{PO} + \vec{OD} = \frac{2}{3}(\vec{PO} + \vec{OR}) + \frac{1}{3}(\vec{PO} + \vec{OM})$

$\cancel{-\vec{p}} + \vec{d} = \cancel{-\frac{2}{3}\vec{p}} + \frac{2}{3}\vec{r} - \cancel{\frac{1}{3}\vec{p}} + \frac{1}{3}\vec{m}$

$\vec{d} = \frac{2}{3}\vec{r} + \frac{1}{3}\left(\dfrac{\vec{p}+\vec{q}}{2}\right)$

$= \frac{1}{6}\vec{p} + \frac{1}{6}\vec{q} + \frac{2}{3}\vec{r}$

16. In ΔABC, M is a point on AB such that $AM : MB = 2 : 3$. N is the mid-point of BC. O is a point on AN such that $AO : ON = 4 : 3$. Let $\mathbf{AB} = \mathbf{p}$ and $\mathbf{AC} = \mathbf{q}$.
    (a) Express $\mathbf{CO}$ in terms of **p** and **q**.
    (b) Are C, O and M collinear? Explain your answer.



(a) $\vec{CA} = -\vec{q}$

$\vec{CB} = \vec{CA} + \vec{AB} = -\vec{q} + \vec{p}$

$\vec{CN} = \frac{1}{2}\vec{CB} = \frac{1}{2}\vec{p} - \frac{1}{2}\vec{q}$

$\vec{CO} = \dfrac{3\vec{CA} + 4\vec{CN}}{7}$

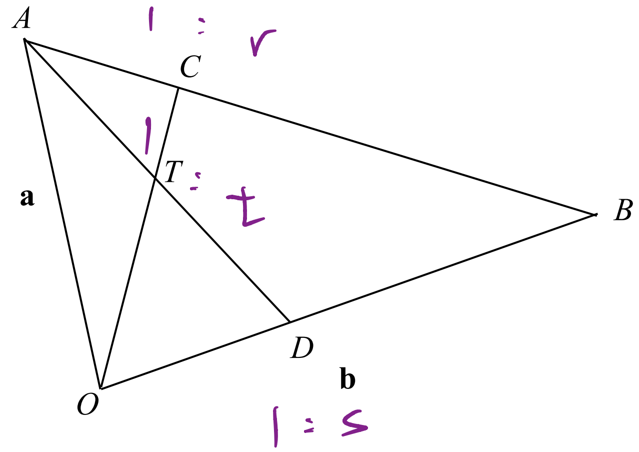
$= -\frac{3}{7}\vec{q} + \frac{4}{7}\left(\frac{1}{2}\vec{p} - \frac{1}{2}\vec{q}\right) = \frac{2}{7}\vec{p} - \frac{5}{7}\vec{q}$

(b) $\vec{CM} = \dfrac{3\vec{CA} + 2\vec{CB}}{5} = -\frac{3}{5}\vec{q} + \frac{2}{5}(-\vec{q} + \vec{p})$

$= \frac{2}{5}\vec{p} - \vec{q}$

$\vec{CO} = \frac{5}{7}\cdot\vec{CM}$, $\therefore$ CO // CM, C, O, M are collinear.

17.



In the figure, $\mathbf{OA}=\mathbf{a}$ and $\mathbf{OB}=\mathbf{b}$. C and D are two points on the lines AB and OB such that

$AC:CB=1:r$, $OD:DB=1:s$ and $AT:TD=1:t$.

(a) Express the vectors $\mathbf{OC}$ and $\mathbf{OT}$ in terms of **a**, **b**, r, s and t.

(b) Express t in terms of r and s.

(a) $\vec{OC}=\dfrac{1\cdot\vec{b}+r\cdot\vec{a}}{1+r}=\dfrac{r}{1+r}\vec{a}+\dfrac{1}{1+r}\vec{b}$

$\vec{OD}=\dfrac{1}{1+s}\vec{OB}=\dfrac{1}{1+s}\vec{b}$

$\vec{OT}=\dfrac{1\cdot\vec{OD}+t\cdot\vec{a}}{1+t}=\dfrac{t}{1+t}\vec{a}+\dfrac{1}{(1+s)(1+t)}\vec{b}$

(b) $\because$ O, C, T are collinear.

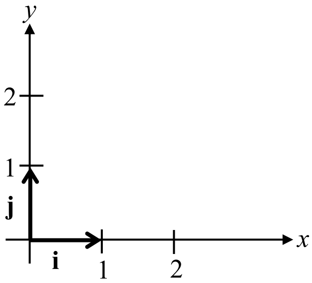
$$\frac{\frac{t}{1+t}}{\frac{r}{1+r}}=\frac{\frac{1}{(1+s)(1+t)}}{\frac{1}{1+r}}$$

$$\frac{t}{r}=\frac{1}{1+s}$$

$$t=\frac{r}{1+s}$$

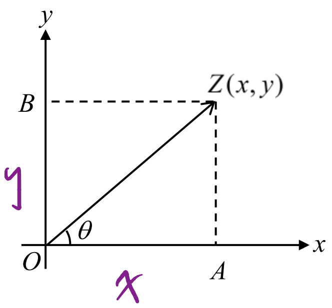
12.4 Vectors in the Rectangular Coordinate System

A. Vectors in a Plane



The two-dimensional vector system consists of two perpendicular unit vectors **i** and **j**.

magnitude = 1



If the origin O of the two-dimensional rectangular coordinate plane is taken as the reference point for position vectors, then any position vector can be expressed in terms of unit vectors **i** and **j.**

(a) $\overrightarrow{OZ} = x\mathbf{i} + y\mathbf{j}$

(b) $\left|\overrightarrow{OZ}\right| = \sqrt{x^2 + y^2}$

(c) $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$ and $\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$

If $\overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\overrightarrow{OB} = x_2\mathbf{i} + y_2\mathbf{j}$, then

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$\overrightarrow{AB} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$ and $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$\overrightarrow{AO} + \overrightarrow{OB}$

Example

15. Find **PQ** in terms of **i** and **j** for the points $P(-6, 2)$ and $Q(-4, -8)$.

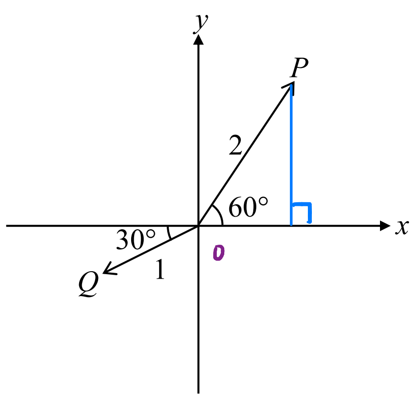
$\vec{OP} = -6\vec{i} + 2\vec{j}$

$\vec{OQ} = -4\vec{i} - 8\vec{j}$

$\vec{PQ} = (-4+6)\vec{i} + (-8-2)\vec{j} = \vec{PO} + \vec{OQ}$

$= 2\vec{i} - 10\vec{j}$

16. Express the position vectors of P and Q in terms of **i** and **j**.



$\vec{OP} = 2\cos 60^\circ\,\vec{i} + 2\sin 60^\circ\,\vec{j}$

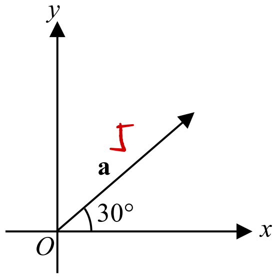
$= \vec{i} + \sqrt{3}\,\vec{j}$

$\vec{OQ} = -1\cos 30^\circ\,\vec{i} - 1\sin 30^\circ\,\vec{j}$

$= -\frac{\sqrt{3}}{2}\vec{i} - \frac{1}{2}\vec{j}$

17. In the figure, $\mathbf{a}$ is a vector making an angle of 30° with the positive x-axis and $|\mathbf{a}|=5$.
    (a) Express $\mathbf{a}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.
    (b) If $\mathbf{b}$ is a vector in the same direction of $\mathbf{a}$ and $|\mathbf{b}|=6$, express $\mathbf{b}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

Hints for (b):
Find the unit vector of $\mathbf{a}$.
Unit vector of $\mathbf{a} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$



(a) $\vec{a} = 5\cos 30^\circ \vec{i} + 5\sin 30^\circ \vec{j} = \frac{5\sqrt{3}}{2}\vec{i} + \frac{5}{2}\vec{j}$

(b) $\frac{1}{5}\vec{a} \times 6 = \frac{6}{5}\vec{a} = 3\sqrt{3}\vec{i} + 3\vec{j}$

$|\vec{a}|$

18. It is given that $\overrightarrow{OA} = 2\mathbf{i}+5\mathbf{j}$ and $\overrightarrow{OB} = -2\mathbf{i}+8\mathbf{j}$.

    (a) Find the vector $\overrightarrow{AB}$.
    (b) Find the unit vector along the direction $\overrightarrow{AB}$.

(a) $\overrightarrow{AB} = -4\vec{i} + 3\vec{j}$

(b) required vector

$= \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$

$= \dfrac{1}{\sqrt{(-4)^2+3^2}} \cdot \overrightarrow{AB}$

$= \frac{1}{5}\overrightarrow{AB}$

$= -\frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$

19. It is given that $\mathbf{a}=2\mathbf{i}+(\lambda-4)\mathbf{j}$ and $\mathbf{b}=3\mathbf{i}+\lambda\mathbf{j}$, where λ is a scalar. If **a** and **b** are parallel, find the value of λ.

$$\frac{3}{2}=\frac{\lambda}{\lambda-4}$$

$$3\lambda-12=2\lambda$$

$$\lambda=12$$

$$\vec{a}=2\vec{i}+8\vec{j}$$

$$\vec{b}=3\vec{i}+12\vec{j}$$

$$\frac{3}{2}\vec{a}=\vec{b}$$

20. It is given that $\mathbf{OA}=\mathbf{i}+\mathbf{j}$, $\mathbf{OB}=2\mathbf{i}+3\mathbf{j}$ and $\mathbf{OC}=k\mathbf{i}+2\mathbf{j}$.

(a) Find **AB** and express **BC** in terms of *k*.

(b) Find the value of *k* if *A*, *B* and *C* are collinear.

(a) $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}=\vec{i}+2\vec{j}$

$\overrightarrow{BC}=\overrightarrow{BO}+\overrightarrow{OC}=(k-2)\vec{i}-\vec{j}$

(b) $AB \parallel BC$

$$\frac{k-2}{1}=\frac{-1}{2}$$

$$2k-4=-1$$

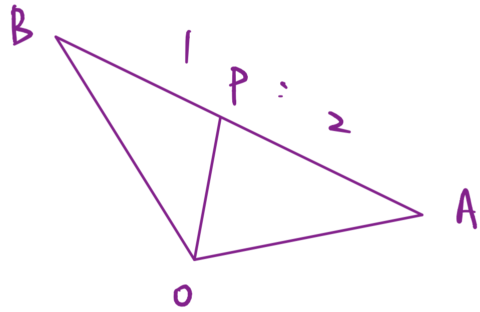
$$k=\frac{3}{2}$$

21. It is given that $\mathbf{OA} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{OB} = -7\mathbf{i} + 3\mathbf{j}$. If P is a point on AB such that $AP : PB = 1 : 2$, find $\mathbf{OP}$.

$$\vec{op} = \frac{1 \cdot \vec{oA} + 2 \cdot \vec{oB}}{1+2}$$

$$= \frac{1}{3}(3\vec{i} + 2\vec{j}) + \frac{2}{3}(-7\vec{i} + 3\vec{j})$$

$$= -\frac{11}{3}\vec{i} + \frac{8}{3}\vec{j}$$



22. It is given that $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 10\mathbf{i} - \mathbf{j}$.

(a) Find $|\mathbf{b} - \mathbf{a}|$.

(b) If a vector $\mathbf{c}$ is opposite to $\mathbf{b} - \mathbf{a}$ and $|\mathbf{c}| = 15$, express $\mathbf{c}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

$\times(-1)$

(a) $\vec{b} - \vec{a}$

$= 8\vec{i} - 6\vec{j}$

$|\vec{b} - \vec{a}|$

$= \sqrt{8^2 + (-6)^2}$

$= 10$

(b) $\vec{c}$

$= -\frac{1}{10}(\vec{b} - \vec{a}) \times 15$

$= -\frac{3}{2}(8\vec{i} - 6\vec{j})$

$= -12\vec{i} + 9\vec{j}$

23. It is given that $\mathbf{u} = -6\mathbf{i} + 9\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} - 5\mathbf{j}$.

(a) Find $|\mathbf{u}+\mathbf{v}|$.

(b) Find the angle between $\mathbf{u}+\mathbf{v}$ and the positive x-axis correct to the nearest $0.1°$.

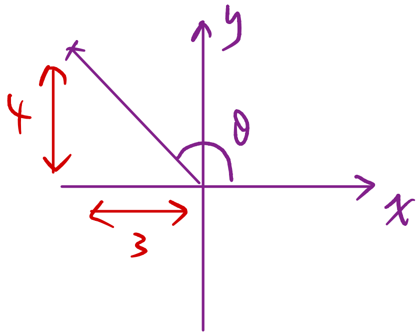
(c) If a vector $\mathbf{p}$ has the same direction as $\mathbf{u}+\mathbf{v}$ and $|\mathbf{p}| = 13$, express $\mathbf{p}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(a) $\vec{u} + \vec{v} = -3\vec{i} + 4\vec{j}$

$|\vec{u}+\vec{v}| = \sqrt{(-3)^2 + 4^2} = 5$

(b) required angle $= 180° - \tan^{-1}\frac{4}{3}$

$= 126.9°$



(c). $\vec{p} = \frac{13}{5}(\vec{u}+\vec{v}) = \frac{-39}{5}\vec{i} + \frac{52}{5}\vec{j}$

24. $A(1,-4)$ and $B(-3,-2)$ are two given points.

(a) Find $|\overrightarrow{AB}|$.

(b) Find the angle between $\overrightarrow{AB}$ and the positive x-axis measured in anticlockwise direction. (Give your answer correct to the nearest $0.1°$)

(c) $\overrightarrow{OC}$ is a vector which is in the direction of $\overrightarrow{AB}$ and $|\overrightarrow{OC}| = \sqrt{5}$.

Find the coordinates of C. $\overrightarrow{OC}$

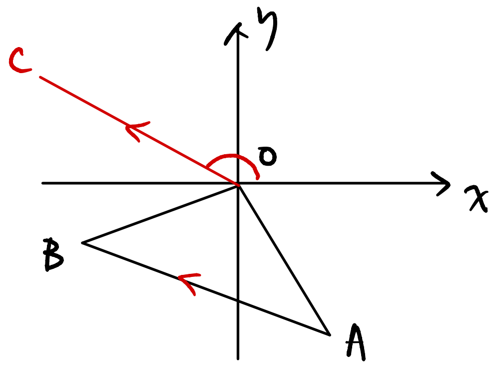
$\overrightarrow{OA} = \vec{i} - 4\vec{j}$ $\quad \overrightarrow{OB} = -3\vec{i} - 2\vec{j}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$

(a) $\overrightarrow{AB} = (-3-1)\vec{i} + (-2-(-4))\vec{j}$

$= -4\vec{i} + 2\vec{j}$

$|\overrightarrow{AB}| = \sqrt{(-4)^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$



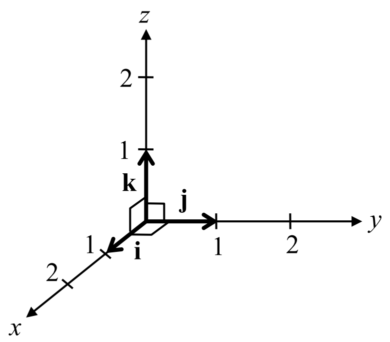
(b) required angle $= 180° - \tan^{-1}\frac{2}{4}$

$= 153.4°$

(c) $\overrightarrow{OC} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} \times \sqrt{5} = \frac{1}{2}\overrightarrow{AB} = -2\vec{i} + \vec{j}$

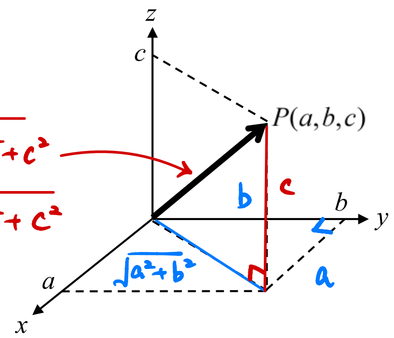
$\therefore C = (-2, 1)$

B. Vectors in Space



$\sqrt{(\sqrt{a^2+b^2})^2+c^2}$

$= \sqrt{a^2+b^2+c^2}$



(a) The three-dimensional vector system consists of three mutually perpendicular unit vectors **i**, **j** and **k**.

(b) The vector **OP** with starting point O and terminal point $P(a,b,c)$ can be represented in the form $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$.

(c) a, b and c are called the x-component, y-component and z-component of **OP**.

(d) If $\mathbf{OP}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, then $|\mathbf{OP}|=\sqrt{a^2+b^2+c^2}$.

Complete the following table.

	Coordinates of P	$\overrightarrow{OP}$	$\lvert\overrightarrow{OP}\rvert$
E.g.	(1, –2, 2)	$\mathbf{i}-2\mathbf{j}+2\mathbf{k}$	$\sqrt{(1)^2+(-2)^2+(2)^2}=3$
(a)	(-2, 3, 6)	$-2\vec{i}+3\vec{j}+6\vec{k}$	$\sqrt{(-2)^2+3^2+6^2}=7$
(b)	(6, -2, -9)	$6\vec{i}-2\vec{j}-9\vec{k}$	$\sqrt{6^2+(-2)^2+(-9)^2}=11$
(c)	(4, -1, 8)	$4\vec{i}-\vec{j}+8\vec{k}$	$\sqrt{4^2+(-1)^2+8^2}=9$

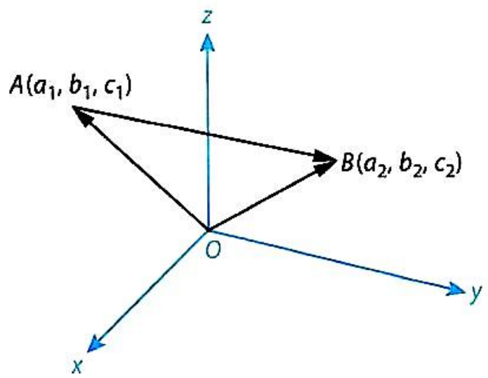
If $\overrightarrow{OA}=a_1\mathbf{i}+b_1\mathbf{j}+c_1\mathbf{k}$ and $\overrightarrow{OB}=a_2\mathbf{i}+b_2\mathbf{j}+c_2\mathbf{k}$, then

$\overrightarrow{AB} = \overrightarrow{AO}+\overrightarrow{OB}$

$= \overrightarrow{OB}-\overrightarrow{OA}$

$= (a_2-a_1)\vec{i}+(b_2-b_1)\vec{j}+(c_2-c_1)\vec{k}$

$|\overrightarrow{AB}| = \sqrt{(a_2-a_1)^2+(b_2-b_1)^2+(c_2-c_1)^2}$



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Example

25. Consider two points $A(1,-2,\ 3)$ and $B(4,\ 1,\ 0)$.

(a) Find $\left|\overrightarrow{AB}\right|$.

(b) Find the unit vector which has the same direction as $\overrightarrow{AB}$.

(a) $\overrightarrow{AB} = 3\vec{i} + 3\vec{j} - 3\vec{k}$

$|\overrightarrow{AB}| = \sqrt{3^2 + 3^2 + 3^2} = 3\sqrt{3}$

(b) $\frac{1}{3\sqrt{3}}(3\vec{i} + 3\vec{j} - 3\vec{k})$

$= \frac{\sqrt{3}}{9}(3\vec{i} + 3\vec{j} + 3\vec{k})$

$= \frac{\sqrt{3}}{3}\vec{i} + \frac{\sqrt{3}}{3}\vec{j} + \frac{\sqrt{3}}{3}\vec{k}$

26. Consider two points $P(-2,3,1)$ and $Q(1,0,-5)$.

(a) $|\mathbf{PQ}|$.

(b) Find the vector $\mathbf{RS}$ which has the same direction as $\mathbf{PQ}$ with $|\mathbf{RS}| = \sqrt{6}$.

(a) $\overrightarrow{PQ} = 3\vec{i} - 3\vec{j} - 6\vec{k}$

$|\overrightarrow{PQ}| = \sqrt{3^2 + (-3)^2 + (-6)^2} = 3\sqrt{6}$

(b) $\overrightarrow{RS} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} \times \sqrt{6}$

$= \frac{1}{3}\overrightarrow{PQ}$

$= \vec{i} - \vec{j} - 2\vec{k}$

Let $\mathbf{p}=a_1\mathbf{i}+b_1\mathbf{j}+c_1\mathbf{k}$ and $\mathbf{q}=a_2\mathbf{i}+b_2\mathbf{j}+c_2\mathbf{k}$ be two non-zero vectors. Then

(i) $\mathbf{p}//\mathbf{q}$ if and only if there exists $m\neq 0$ such that $\mathbf{p}=m\mathbf{q}$.

(ii) $\mathbf{p}//\mathbf{q}$ if and only if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

Example

27. Four points $A(1,\ 3,\ 5)$, $B(2,\ 6,\ r)$, $C(1,\ m,-2)$ and $D(2,\ 5,\ 8)$ are given. If $AB//CD$, find the values of m and r.

$\vec{AB}=(2-1)\vec{i}+(6-3)\vec{j}+(r-5)\vec{k}$

$=\vec{i}+3\vec{j}+(r-5)\vec{k}$

$\vec{CD}=(2-1)\vec{i}+(5-m)\vec{j}+(8+2)\vec{k}$

$=\vec{i}+(5-m)\vec{j}+10\vec{k}$

$3=5-m$

$m=2$

$r-5=10$

$r=15$

28. Three points $A(5,\ 2,\ 7)$, $B(2,\ 1,\ 3)$ and $C(8,\ 3,\ 11)$ are given. Determine whether A, B and C are collinear.

$\vec{AB}=-3\vec{i}-\vec{j}-4\vec{k}$

$\vec{BC}=6\vec{i}+2\vec{j}+8\vec{k}$

$-2\vec{AB}=\vec{BC}$

$\frac{\vec{AB}}{\vec{BC}}=-\frac{1}{2}$ X

29. It is given that the points $P(1,\ m-7,\ n)$, $Q(0,\ m-2,\ 5)$ and $R(2,\ 17,\ 5n-2)$ are collinear. Find the values of m and n.

$\vec{PR}=\vec{i}+(24-m)\vec{j}+(4n-2)\vec{k}$

$\vec{QR}=2\vec{i}+(19-m)\vec{j}+(5n-7)\vec{k}$

$\frac{1}{2}=\frac{24-m}{19-m}$

$19-m=48-2m$

$m=29$

$\frac{1}{2}=\frac{4n-2}{5n-7}$

$5n-7=8n-4$

$n=-1$