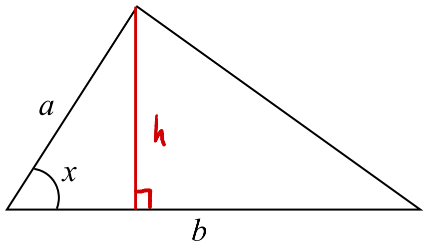
Chapter 3 More About Trigonometric Functions (3.4 – 3.6)
Supplementary Notes

Name: ________________________________ (   )   Class: F.4 _____

Finding Area of a Triangle   2 sides + included ∠

Area of the triangle $= \frac{1}{2}ab\sin x$



$\sin x = \frac{h}{a}$ area $= \frac{1}{2} \cdot b \cdot h$

$h = a\sin x$ $= \frac{1}{2} \cdot b \cdot a\sin x$

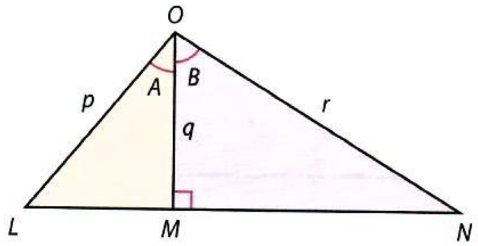
3.4 Compound Angle Formulae $= \frac{1}{2} \cdot a \cdot b \cdot \sin x$

Class Activity

The figure shows ΔOLN, where M is a point lying on LN such that $\angle OMN$ is a right angle.

Let $\angle LOM = A$ and $\angle MON = B$, then $\angle LON = A + B$.

1. Express each of the following in terms of p, q, r, A and B.



(a) Area of ΔLOM = $\frac{1}{2} \cdot p \cdot q \cdot \sin A$

(b) Area of ΔMON = $\frac{1}{2} \cdot q \cdot r \cdot \sin B$

(c) Area of ΔLON = $\frac{1}{2} \cdot p \cdot r \cdot \sin(A+B)$

2. Consider 'Area of ΔLOM + Area of ΔMON = Area of ΔLON'

$$\frac{\cancel{\frac{1}{2}} pq\sin A + \cancel{\frac{1}{2}} qr\sin B}{pr} = \frac{\cancel{\frac{1}{2}} pr\sin(A+B)}{pr}$$

$\sin(A+B) = \frac{q}{r}\sin A + \frac{q}{p}\sin B$ $\Rightarrow$ $\sin(A+B) = \sin A\cos B + \cos A\sin B$

3. Express $\frac{q}{r}$ and $\frac{q}{p}$ in terms of trigonometric ratios.

$\frac{q}{r} = \cos B$ $\frac{q}{p} = \cos A.$

Compound Angle Formulae

$\sin(A+B)=\sin A\cos B+\cos A\sin B$ $\quad$ $\sin(A-B)=\sin A\cos B-\cos A\sin B$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$ $\quad$ $\cos(A-B)=\cos A\cos B+\sin A\sin B$

$\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ $\quad$ $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

	$30^\circ\left(\frac{\pi}{6}\right)$	$45^\circ\left(\frac{\pi}{4}\right)$	$60^\circ\left(\frac{\pi}{3}\right)$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}\right)$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{3}\right)$	1	$\sqrt{3}$

Example

1. Find the value of $\sin 15^\circ$ in surd form.

$\sin 15^\circ$

$= \sin(60^\circ - 45^\circ)$

$= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$

$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

2. Find the value of $\cos 75^\circ$ in surd form.

$\cos 75^\circ$

$= \cos(30^\circ + 45^\circ)$

$= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$

$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$

$= \dfrac{\sqrt{6}-\sqrt{2}}{4}$

OR

$\cos 75^\circ$

$= \cos(90^\circ - 15^\circ)$

$= \sin 15^\circ$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\left(\dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}\right)}{\left(\dfrac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}\right)}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$\frac{7\pi}{12} = \frac{7}{12} \times 180^\circ = 105^\circ = 60^\circ + 45^\circ$

3. Find the value of $\tan\left(\frac{7\pi}{12}\right)$ in surd form.

$\tan\left(\frac{7\pi}{12}\right)$

$= \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$= \frac{\tan\frac{\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{3} \cdot \tan\frac{\pi}{4}}$

$= \frac{\sqrt{3}+1}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}}$

$= \frac{(\sqrt{3}+1)^2}{1-3}$

$= \frac{3+2\sqrt{3}+1}{-2}$

$= \frac{4+2\sqrt{3}}{-2}$

$= -2-\sqrt{3}$

Rationalization of denominator

$$\frac{1}{1+\sqrt{a}} = \frac{1}{1+\sqrt{a}} \cdot \frac{1-\sqrt{a}}{1-\sqrt{a}}$$

$$= \frac{1-\sqrt{a}}{1^2-(\sqrt{a})^2}$$

$$= \frac{1-\sqrt{a}}{1-a}$$

$$\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{\sqrt{a}+\sqrt{b}} \cdot \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$$

$$= \frac{\sqrt{a}-\sqrt{b}}{(\sqrt{a})^2-(\sqrt{b})^2}$$

$$= \frac{\sqrt{a}-\sqrt{b}}{a-b}$$

4. Simplify $\underbrace{\cos(x-y)}_{A}\underbrace{\cos y}_{B}-\underbrace{\sin(x-y)}_{A}\underbrace{\sin y}_{B}$.

$= \cos(x-y+y)$

$= \cos x$

5. Prove the following.

(a) $\sin(A+B)\cos B-\cos(A+B)\sin B-\sin A=0$

(b) $\tan\left(A-\frac{\pi}{4}\right)=\frac{\sin A-\cos A}{\sin A+\cos A}$

(a) L.H.S.

$= \sin(A+B-B) - \sin A$

$= \sin A - \sin A$

$= 0$

$=$ R.H.S.

(b). L.H.S.

$= \frac{\tan A - \tan\frac{\pi}{4}}{1+\tan A\tan\frac{\pi}{4}}$

$= \frac{\tan A - 1}{1+\tan A}$

$= \frac{\frac{\sin A}{\cos A}-1}{1+\frac{\sin A}{\cos A}}$

$= \frac{\sin A - \cos A}{\cos A} \times \frac{\cos A}{\cos A + \sin A}$.

$= \frac{\sin A - \cos A}{\sin A + \cos A}$.

$=$ R.H.S.

6. Let $\sin(x+\alpha) = 3\cos(x-\alpha)$, where $\cos x \neq 0$ and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

(a) Prove that $\tan x = \frac{3-\tan\alpha}{1-3\tan\alpha}$.

(b) Hence solve the equation $\sin\left(\theta - \frac{\pi}{4}\right) = 3\cos\left(\theta + \frac{\pi}{4}\right)$, where $0 \le \theta \le 2\pi$.

Hint of (b): Consider the equation.

$x =$ θ , $\alpha =$ $-\frac{\pi}{4}$

(a) $\sin(x+\alpha) = 3\cos(x-\alpha)$

$$\frac{\sin x\cos\alpha + \cos x\sin\alpha}{\cos x\cos\alpha} = \frac{3\cos x\cos\alpha + 3\sin x\sin\alpha}{\cos x\cos\alpha}$$

$$\tan x + \tan\alpha = 3 + 3\tan x\tan\alpha$$

$$\tan x - 3\tan x\tan\alpha = 3 - \tan\alpha$$

$$\tan x(1 - 3\tan\alpha) = 3 - \tan\alpha$$

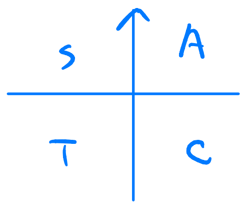
$$\tan x = \frac{3-\tan\alpha}{1-3\tan\alpha}$$

(b) put $x = \theta$, $\alpha = -\frac{\pi}{4}$ into (a)

$$\tan\theta = \frac{3-\tan(-\frac{\pi}{4})}{1-3\tan(-\frac{\pi}{4})} = \frac{3-(-1)}{1-(-3)} = 1$$

$$\theta = \frac{\pi}{4} \text{ or } \pi + \frac{\pi}{4}$$

$$= \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$



3.5 Double Angle Formulae

By considering the compound angle formulae, express each of the following in terms of $\sin A$ and $\cos A$.

1. $\sin 2A$

$= \sin(A+A)$

$= \sin A \cos A + \cos A \sin A$

$= 2\sin A \cos A$.

2. $\cos 2A$

$= \cos(A+A)$

$= \cos A \cos A - \sin A \cdot \sin A$

$= \cos^2 A - \sin^2 A$

$= \cos^2 A - (1 - \cos^2 A)$

$= 2\cos^2 A - 1$

$= 1 - \sin^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

3. $\tan 2A$

$= \tan(A+A)$

$= \dfrac{\tan A + \tan A}{1 - \tan A \cdot \tan A}$

$= \dfrac{2\tan A}{1 - \tan^2 A}$

Double Angle Formulae

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\quad = 2\cos^2 A - 1$

$\quad = 1 - 2\sin^2 A$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

We can also express $\cos^2 A$ and $\sin^2 A$ in terms of trigonometric ratios by the double angles.

$\boxed{\cos^2 A = \frac{1}{2}(1+\cos 2A)}$ and $\boxed{\sin^2 A = \frac{1}{2}(1-\cos 2A)}$

- These two formulae can be used to **reduce the powers of** trigonometric ratios.

Example

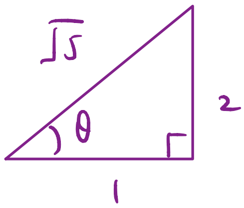
7. It is given that $\sin\theta = \frac{2}{\sqrt{5}}$ and θ is an acute angle. Find the value of each of the following without finding θ.

(a) $\cos\theta$ and $\tan\theta$

(b) $\sin 2\theta$

(c) $\cos 2\theta$

(d) $\tan 2\theta$



(a) $\cos\theta = \frac{1}{\sqrt{5}}$, $\tan\theta = 2$

(b) $\sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2\,\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}$

$= \frac{4}{5}$

(c) $\cos 2\theta$

$= \cos^2\theta - \sin^2\theta$

$= \frac{1}{5} - \frac{4}{5}$

$= -\frac{3}{5}$

(d) $\tan 2\theta$

$= \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\cdot 2}{1-2^2}$

$= -\frac{4}{3}$

$2A \to A$

* 8. (a) Prove that $\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$.

(b) Hence, solve $\frac{1-\cos 2A}{1+\cos 2A} = 2\tan A - 1$ for $0 \le A \le 2\pi$.

(a) L.H.S. $= \frac{1-(1-2\sin^2 A)}{1+2\cos^2 A - 1}$ ← remove 1 in denominator and numerator

$= \frac{2\sin^2 A}{2\cos^2 A}$

$= \tan^2 A$

$=$ R.H.S

(b) $\tan^2 A = 2\tan A - 1$

$\tan^2 A - 2\tan A + 1 = 0$

$(\tan A - 1)^2 = 0$

$\tan A = 1$

$A = \frac{\pi}{4}$ or $\pi + \frac{\pi}{4}$

$A = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1$

9. (a) Prove that $\dfrac{\sin\theta}{1-\cos\theta} = \cot\dfrac{\theta}{2}$.

(b) Hence, find $\cot\dfrac{\pi}{8}$.

$\dfrac{\pi}{8} = \dfrac{\theta}{2}$

$\theta = \dfrac{\pi}{4}$

Hint:

$\sin 2\theta = 2\sin\theta\cos\theta \rightarrow$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\cos 2\theta = 1-2\sin^2\theta \rightarrow$

$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

(a) $$\text{L.H.S.} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-(1-2\sin^2\frac{\theta}{2})} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

(b) $$\cot\frac{\pi}{8} = \frac{\sin\frac{\pi}{4}}{1-\cos\frac{\pi}{4}} = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$$

✓ $$= \frac{1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$$

10. (a) Prove that $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$.

(b) Hence, prove that $\dfrac{\sin 4y(1-\cos 2y)}{\cos 2y(1-\cos 4y)} = \tan y$.

(a) $$\text{L.H.S.} = \frac{1-(1-2\sin^2 x)}{2\sin x\cos x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x$$

(b) $$\text{L.H.S.} = \frac{\sin 4y(1-\cos 2y)}{\cos 2y(1-\cos 4y)} = \frac{\frac{1}{\tan 2y}(1-\cos 2y)}{\cos 2y} = \frac{\frac{\cos 2y}{\sin 2y}(1-\cos 2y)}{\cos 2y} = \frac{1-\cos 2y}{\sin 2y} = \tan y$$

* 11. Prove that Triple Angle Formula

(a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, 有 $\cos\theta$

L.H.S.

$= \sin 3\theta$

$= \sin(2\theta + \theta)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

(b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. 有 $\sin\theta$

L.H.S.

$= \cos 3\theta$

$= \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

12. Prove that $\tan A \tan 2A + 1 = \sec 2A$.

L.H.S. $= \tan A \cdot \dfrac{2\tan A}{1 - \tan^2 A} + 1$

$= \dfrac{2\tan^2 A + 1 - \tan^2 A}{1 - \tan^2 A}$

$= \dfrac{1 + \tan^2 A}{1 - \tan^2 A}$

$= \dfrac{\sec^2 A}{1 - \tan^2 A}$

$= \dfrac{1}{\cos^2 A(1 - \tan^2 A)}$

$= \dfrac{1}{\cos^2 A - \sin^2 A}$

$= \dfrac{1}{\cos 2A}$

R.H.S. $= \dfrac{1}{\cos 2A}$

$\tan\theta > 0$

13. Suppose $0 < \theta < \dfrac{\pi}{2}$. If $\tan 2\theta = \dfrac{3}{4}$, find the value of $\tan\theta$ without finding the value of θ.

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{3}{4}$$

$$8\tan\theta = 3 - 3\tan^2\theta$$

$$3\tan^2\theta + 8\tan\theta - 3 = 0$$

$$(3\tan\theta - 1)(\tan\theta + 3) = 0$$

$$\tan\theta = \frac{1}{3} \text{ or } \tan\theta = -3 \text{ (rej.)}$$

Let $y = \tan\theta$

$$3y^2 + 8y - 3 = 0$$

$$(3y-1)(y+3) = 0$$

$$y = \frac{1}{3} \text{ or } -3$$

14. It is given that $\cos\theta \neq -\dfrac{1}{2}$ and n is a positive integer.

(a) Prove that $2\cos\theta - 1 = \dfrac{2\cos 2\theta + 1}{2\cos\theta + 1}$.

$2\cos 2\theta - 1 = \dfrac{2\cos 4\theta + 1}{2\cos 2\theta + 1}$ $\quad$ $2\cos 4\theta - 1 = \dfrac{2\cos 8\theta + 1}{2\cos 4\theta + 1}$

(b) Hence, prove that $(2\cos\theta - 1)(2\cos 2\theta - 1)\cdots[2\cos(2^{n-1}\theta) - 1)] = \dfrac{2\cos(2^n\theta)+1}{2\cos\theta + 1}$.

Cancellation

(a) R.H.S.

$$= \frac{2\cos 2\theta + 1}{2\cos\theta + 1}$$

$$= \frac{2(2\cos^2\theta - 1) + 1}{2\cos\theta + 1}$$

$$= \frac{4\cos^2\theta - 1}{2\cos\theta + 1}$$

$$= \frac{(2\cos\theta + 1)(2\cos\theta - 1)}{2\cos\theta + 1}$$

$$= 2\cos\theta - 1$$

(b) L.H.S.

$$= \frac{2\cos 2\theta + 1}{2\cos\theta + 1} \cdot \frac{2\cos 4\theta + 1}{2\cos 2\theta + 1} \cdot \frac{2\cos 8\theta + 1}{2\cos 4\theta + 1} \cdots$$

$$\frac{2\cos(2^n\theta) + 1}{2\cos(2^{n-1}\theta) + 1}$$

$$= \frac{2\cos(2^n\theta) + 1}{2\cos\theta + 1}$$

3.6 Sum and Product Formula

A. Product to Sum Formulae

$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$

$$2\cos A\cos B = \cos(A+B)+\cos(A-B)$$

$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$

B. Sum to Product Formulae

$$\sin A+\sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A-\sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A+\cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A-\cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Example

$\cos(-\theta)=\cos\theta$

$\cos(-36^\circ)=\cos 36^\circ$

15. Express each of the following as a sum or a difference of sine and cosine.

(a) $\sin 48^\circ\cos 32^\circ$

$= \frac{1}{2}[\sin(48^\circ+32^\circ)+\sin(48^\circ-32^\circ)]$

$= \frac{1}{2}(\sin 80^\circ+\sin 16^\circ)$

(b) $\sin 32^\circ\sin 68^\circ$

$= -\frac{1}{2}[\cos(32^\circ+68^\circ)-\cos(32^\circ-68^\circ)]$

$= -\frac{1}{2}(\cos 100^\circ-\cos 36^\circ)$

(c) $2\cos\frac{4\pi}{5}\cos\frac{3\pi}{5}$

$= 2\cdot\frac{1}{2}\left[\cos\left(\frac{4\pi}{5}+\frac{3\pi}{5}\right)+\cos\left(\frac{4\pi}{5}-\frac{3\pi}{5}\right)\right]$

$= \cos\frac{7\pi}{5}+\cos\frac{\pi}{5}$

(d) $\sin(x+y)\sin(x-y)$

$= -\frac{1}{2}[\cos(x+y+x-y)-\cos(\underbrace{x+y-x+y}_{-(x-y)})]$

$= -\frac{1}{2}(\cos 2x-\cos 2y)$

16. Express the following as products of sines/cosines.

(a) $\cos 3A - \cos 2A$

$= -2\sin\left(\frac{3A+2A}{2}\right)\sin\left(\frac{3A-2A}{2}\right)$

(b) $\sin 4A - \sin A$

$= 2\cos\left(\frac{4A+A}{2}\right)\sin\left(\frac{4A-A}{2}\right)$

$= 2\cos\frac{5A}{2}\sin\frac{3A}{2}$

$x+y-(x-y)$

(c) $\cos(x+y)+\cos(x-y)$

$= 2\cos\left(\frac{x+y+x-y}{2}\right)\cos\left(\frac{x+y-x+y}{2}\right)$

$= 2\cos x\cos y$

$\sin(-\theta) = -\sin\theta$

(d) $\cos\frac{\pi}{5} - \cos\frac{3\pi}{5}$

$= -2\sin\left(\frac{\frac{\pi}{5}+\frac{3\pi}{5}}{2}\right)\sin\left(\frac{\frac{\pi}{5}-\frac{3\pi}{5}}{2}\right)$

$= -2\sin\frac{2\pi}{5}\sin\left(-\frac{\pi}{5}\right)$

$= 2\sin\frac{2\pi}{5}\sin\frac{\pi}{5}$

17. Prove that $\frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$.

$\text{L.H.S.} = \frac{2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}{2\cos\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}$

$= \frac{\sin 2x}{\cos 2x}$

$= \tan 2x$

18. Prove that $\dfrac{\sin A+\sin 3A+\sin 5A}{\cos A+\cos 3A+\cos 5A}=\tan 3A$.

Hint:
We first group

$\sin A$ & $\sin 5A$

$\cos A$ & $\cos 5A$

$$\text{L.H.S.} = \frac{\sin 5A + \sin A + \sin 3A}{\cos 5A + \cos A + \cos 3A}$$

$$= \frac{2\sin 3A \cos 2A + \sin 3A}{2\cos 3A \cos 2A + \cos 3A}$$

$$= \frac{\sin 3A\,(2\cos 2A + 1)}{\cos 3A\,(2\cos 2A + 1)}$$

$$= \tan 3A.$$

19. Prove that $\cos 2\theta + 2\cos 4\theta + \cos 6\theta = 4\cos^2\theta\cos 4\theta$.

$$\text{L.H.S.} = 2\cos 4\theta \cos 2\theta + 2\cos 4\theta$$

$$= 2\cos 4\theta\,(\cos 2\theta + 1)$$

$$= 2\cos 4\theta\,(2\cos^2\theta - 1 + 1)$$

$$= 2\cos 4\theta \cdot 2\cos^2\theta$$

$$= 4\cos^2\theta \cos 4\theta$$

20. Solve $\cos x + \cos 2x + \cos 3x = 0$ for $0 \le x \le 2\pi$.

$\cos x + \cos 2x + \cos 3x = 0$

$2\cos\frac{3x}{2}\cos\frac{x}{2} + \cos 3x = 0$ ✗

$\cos 2x + 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) = 0$

$\cos 2x + 2\cos 2x\cos x = 0$

$\cos 2x(1 + 2\cos x) = 0$

$0 \le x \le 2\pi$

$0 \le 2x \le 4\pi$

$\cos 2x = 0$ or $1 + 2\cos x = 0$

$\cos 2x = 0$ or $\cos x = -\frac{1}{2}$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ or $x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$x = \frac{\pi}{4}, \frac{2\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{2\pi}{3}, \frac{4\pi}{3}$

21. Solve $\sin x - \sin 5x + \sin 9x = 0$ for $0 \le x \le \pi$.

$\sin 9x + \sin x - \sin 5x = 0$

$2\sin\left(\frac{9x+x}{2}\right)\cos\left(\frac{9x-x}{2}\right) - \sin 5x = 0$

$2\sin 5x\cos 4x - \sin 5x = 0$

$\sin 5x(2\cos 4x - 1) = 0$

$\sin 5x = 0$ or $\cos 4x = \frac{1}{2}$

$5x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$ or $4x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}$

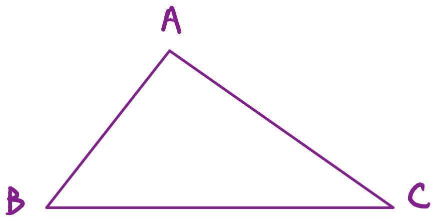
$x = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

22. In ΔABC, it is given that $\cos A+\cos B-\sin C=0$.

(a) Show that $\cos\left(\frac{A+B}{2}\right)=\sin\frac{C}{2}$.

(b) Show that $\cos\left(\frac{A-B}{2}\right)=\cos\frac{C}{2}$.

(c) Is ΔABC a right-angled triangle? Explain your answer.



Hint of (a)

$A+B+C=$ $\underline{180^\circ = \pi}$

(a) $L.H.S = \cos\left(\frac{A+B}{2}\right)$

$= \cos\left(\frac{\pi - C}{2}\right)$

$= \cos\left(\frac{\pi}{2} - \frac{C}{2}\right)$

$= \sin\frac{C}{2}$

(b)

$\cos A + \cos B - \sin C = 0$

$2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \sin C = 0$

$2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2} = 0$

$2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\frac{C}{2}\right] = 0$

$\neq 0$

$\therefore \cos\left(\frac{A-B}{2}\right) - \cos\frac{C}{2} = 0$

$\cos\left(\frac{A-B}{2}\right) = \cos\frac{C}{2}$

(c) $\frac{A-B}{2} = \frac{C}{2}$

$A - B = C$

$A+B+C = \pi$

$A+B+A-B = \pi$

$2A = \pi$

$A = \frac{\pi}{2}$

23. (a) Prove that $\cos 10° \sin 10° \sin 50° \sin 70° = \frac{1}{8}\sin 100°$.

(b) Hence, find the value of $\sin 10° \sin 50° \sin 70°$ without using a calculator.

(a) $\text{L.H.S.} = \sin 10° \cos 10° \sin 50° \sin 70°$

$= \frac{1}{2}\left(\sin 20° + \cancel{\sin 0°}^{\,0}\right) \sin 50° \sin 70°$

$= \frac{1}{2} \sin 20° \sin 50° \sin 70°$

$= \frac{1}{2} \sin 50° \cdot \left(-\frac{1}{2}\right)\left(\cos 90° - \cos 50°\right)$

$= \frac{1}{4} \sin 50° \cos 50°$

$= \frac{1}{4} \cdot \frac{1}{2} \cdot \left(\sin 100° + \sin 0°\right)$

$= \frac{1}{8} \sin 100°$

(b) $\sin 10° \sin 50° \sin 70° = \dfrac{\frac{1}{8}\sin 100°}{\cos 10°}$

$= \dfrac{\frac{1}{8}\sin(90° + 10°)}{\cos 10°}$

$= \dfrac{\frac{1}{8}\cos 10°}{\cos 10°}$

$= \frac{1}{8}$

$\sin(90° + \theta) = \cos\theta$

without using calculator.

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

24. By considering $\sin\frac{\pi}{9}\left(\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\right)$, find the value of $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$.

$$\underline{\sin\frac{\pi}{9}\cos\frac{\pi}{9}}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$$

$$= \frac{1}{2}\underline{\sin\frac{2\pi}{9}\cos\frac{2\pi}{9}}\cdot\cos\frac{4\pi}{9}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\sin\frac{4\pi}{9}\cdot\cos\frac{4\pi}{9}$$

$$= \frac{1}{4}\cdot\frac{1}{2}\cdot\sin\frac{8\pi}{9}$$

$$= \frac{1}{8}\sin\frac{8\pi}{9}$$

$$\therefore \cos\frac{\pi}{9}\cdot\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{\frac{1}{8}\sin\frac{8\pi}{9}}{\sin\frac{\pi}{9}}$$

$\sin(\pi-\theta) = \sin\theta$

$$= \frac{\frac{1}{8}\sin\left(\pi-\frac{\pi}{9}\right)}{\sin\frac{\pi}{9}}$$

$$= \frac{\frac{1}{8}\sin\frac{\pi}{9}}{\sin\frac{\pi}{9}}$$

$$= \frac{1}{8}$$

25. (a) Prove, by mathematical induction, that

$$\sin 2\theta + \sin 4\theta + \sin 6\theta + \ldots + \sin 2n\theta = \frac{\sin(n+1)\theta \sin n\theta}{\sin\theta},$$ where $\sin\theta \neq 0$, for all positive integers n.

$$\frac{\sin(k+2)\theta \sin(k+1)\theta}{\sin\theta}$$

(b) Using (a) and the substitution $\theta = \frac{\pi}{2} - x$, or otherwise, show that

$$\sin 2x - \sin 4x + \sin 6x = \frac{\sin 4x \cos 3x}{\cos x},$$ where $\cos x \neq 0$.

(a) When $n=1$, L.H.S. $= \sin 2\theta$

R.H.S. $= \dfrac{\sin 2\theta \sin\theta}{\sin\theta} = \sin 2\theta$

$\therefore$ the result holds when $n=1$

Assume the result holds when $n=k$,

$$\sin 2\theta + \sin 4\theta + \sin 6\theta + \cdots + \sin 2k\theta = \frac{\sin(k+1)\theta \sin k\theta}{\sin\theta}$$

When $n=k+1$

$$\text{L.H.S.} = \sin 2\theta + \sin 4\theta + \sin 6\theta + \cdots + \sin 2k\theta + \sin(2k+2)\theta$$

$$= \frac{\sin(k+1)\theta \sin k\theta}{\sin\theta} + \sin(2k+2)\theta$$

$$= \frac{\sin(k+1)\theta \sin k\theta + \sin(2k+2)\theta \sin\theta}{\sin\theta}$$

$$= \frac{-\frac{1}{2}[\cos(2k+1)\theta - \cos\theta] - \frac{1}{2}[\cos(2k+3)\theta - \cos(2k+1)\theta]}{\sin\theta}$$

$$= \frac{\frac{1}{2}[\cos\theta - \cos(2k+3)\theta]}{\sin\theta}$$

$$= \frac{-\frac{1}{2}[\cos(2k+3)\theta - \cos\theta]}{\sin\theta}$$

25. (a) Prove, by mathematical induction, that

$$\sin 2\theta + \sin 4\theta + \sin 6\theta + \ldots + \sin 2n\theta = \frac{\sin(n+1)\theta \sin n\theta}{\sin\theta},$$ where $\sin\theta \neq 0$, for all positive integers n.

(b) Using (a) and the substitution $\theta = \frac{\pi}{2} - x$, or otherwise, show that

$n=3$

$$\sin 2x - \sin 4x + \sin 6x = \frac{\sin 4x \cos 3x}{\cos x},$$ where $\cos x \neq 0$.

(b) $\sin 2\theta + \sin 4\theta + \sin 6\theta = \dfrac{\sin 4\theta \sin 3\theta}{\sin\theta}$

$\sin\left[2\left(\frac{\pi}{2}-x\right)\right] + \sin\left[4\left(\frac{\pi}{2}-x\right)\right] + \sin\left[6\left(\frac{\pi}{2}-x\right)\right] = \dfrac{\sin\left[4\left(\frac{\pi}{2}-x\right)\right]\sin\left[3\left(\frac{\pi}{2}-x\right)\right]}{\sin\left(\frac{\pi}{2}-x\right)}$

$(\pi - 6x)$

$\sin(\pi-2x) + \sin(2\pi-4x) + \sin(3\pi-6x) = \dfrac{\sin(2\pi-4x)\sin\left(\frac{3\pi}{2}-3x\right)}{\sin\left(\frac{\pi}{2}-x\right)}$

$\sin 2x + (-\sin 4x) + \sin 6x = \dfrac{-\sin 4x \cdot (-\cos 3x)}{\cos x}$

$\sin 2x - \sin 4x + \sin 6x = \dfrac{\sin 4x \cos 3x}{\cos x}$

26. (a) Using mathematical induction, prove that $\sum_{k=1}^{n}\sin(2k-1)\theta=\frac{1-\cos 2n\theta}{2\sin\theta}$ for all positive integers n.

$\frac{1-\cos(2m+2)\theta}{2\sin\theta}$

(b) Hence, solve $\sum_{k=1}^{4}\sin(2k-1)x=0$ for $0<x<\pi$.

(a) When $n=1$, L.H.S $=\sin\theta$

R.H.S $=\frac{1-\cos 2\theta}{2\sin\theta}=\frac{1-(1-2\sin^2\theta)}{2\sin\theta}=\frac{2\sin^2\theta}{2\sin\theta}=\sin\theta$

Assume result holds when $n=m$

$\sum_{k=1}^{m}\sin(2k-1)\theta=\frac{1-\cos 2m\theta}{2\sin\theta}$

When $n=m+1$

L.H.S. $=\sum_{k=1}^{m}\sin(2k-1)\theta+\sin\underbrace{(2m+1)}_{2(m+1)-1}\theta$

$=\frac{1-\cos 2m\theta}{2\sin\theta}+\sin(2m+1)\theta$

$=\frac{1-\cos 2m\theta+2\sin(2m+1)\theta\sin\theta}{2\sin\theta}$

$=\frac{1-\cos 2m\theta+2(-\frac{1}{2})[\cos(2m+2)\theta-\cos 2m\theta]}{2\sin\theta}$

$=\frac{1-\cos 2m\theta-\cos(2m+2)\theta+\cos 2m\theta}{2\sin\theta}$

$=\frac{1-\cos(2m+2)\theta}{2\sin\theta}=$ R.H.S.

$\therefore$ the result holds when $n=m+1$

By M.I. the result holds for all positive integer n.

26. (a) Using mathematical induction, prove that $\sum_{k=1}^{n}\sin(2k-1)\theta=\frac{1-\cos 2n\theta}{2\sin\theta}$ for all positive integers n.

$n=4,\ \theta=x$

(b) Hence, solve $\sum_{k=1}^{4}\sin(2k-1)x=0$ for $0<x<\pi$.

← does not include $0,\ \pi$.

(b) $\sum_{k=1}^{4}\sin(2k-1)x=0$

$\frac{1-\cos 8x}{2\sin x}=0$

$\sin\theta\neq 0 \rightarrow 2\sin x$

$1-\cos 8x=0$

$\cos 8x=1$

$0<x<\pi$

$0<8x<8\pi$

$8x=2\pi,\ 4\pi,\ 6\pi$

$x=\frac{\pi}{4},\ \frac{\pi}{2},\ \frac{3\pi}{4}$

Summary

Special Angles

	$30°\ \left(\frac{\pi}{6}\right)$	$45°\ \left(\frac{\pi}{4}\right)$	$60°\ \left(\frac{\pi}{3}\right)$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}\ \left(\frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}\ \left(\frac{\sqrt{2}}{2}\right)$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}\ \left(\frac{\sqrt{3}}{3}\right)$	1	$\sqrt{3}$

Given in the Public Exam

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Double Angle Formulae

$$\sin 2A = 2\sin A \cos A$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\cos^2 A = \frac{1}{2}(1+\cos 2A),\quad \sin^2 A = \frac{1}{2}(1-\cos 2A)$$